NONSYMMETRIC CORRESPONDENCE ANALYSIS
for THREE-WAY CONTINGENCY TABLES

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Abstract

The problem of studying the structure of dependence among three qualitative variables which play a different role in the analysis is approached by means Nonsymmetric Correspondence Analysis for three-way contingency table (NSC3). The aim of this work is to discuss the interpretation of the technique and the tridimensional measure of dependence, on which it is based (τ, Marcotorchino 1984). Nonsymmetric Correspondence Analysis for three-way contingency tables is presented using the Tucker3 model (Tucker 1966) in combination with Lancaster's (1951) additive decomposition of interactions in three-way tables. An example is presented to illustrate the theoretical developments.

Key words: *Nonsymmetric measure of dependence, Nonsymmetric Correspondence Analysis, Orthogonal decomposition, Tucker3, Biplot....*

1 Introduction

Within the dependence framework with two categorical variables where some variables are logically antecedent to the others (Graphical models, Whittaker 1990; Canonical Correspondence Analysis, Ter Braak 1986, Sabatier, Lebreton and Chessel 1988; Redundancy Analysis, Israël 1984; Logit analysis, Bishop, Fienberg and Holland 1975 etc...) Nonsymmetric Correspondence Analysis (NSCA, Lauro & D'Ambra 1984; D'Ambra & Lauro 1989, 1992) allows to assess, in an exploratory fashion, the strength of the dependence of the two categorical variables and to describe and portray the nature of the dependence.
The main aim of the present paper is to extend Nonsymmetric Correspondence Analysis to three-way tables. A parallel to the nonsymmetric case is three-way Correspondence Analysis (Carlier & Kroonenberg 1994) which extends Correspondence Analysis to three-way tables using three-way generalisations of the Singular Value Decomposition (SVD, Eckart & Young 1936).
Previous paper on extending two-way NSCA, treated three-way tables reducing them to two-way tables (D'Ambra & Lauro 1989), in this paper three-way contingency tables are analysed without collapsing them in two-way form. Particular attention will be paid to three-way generalizations of SVD and precisely to the Tucker3 model (Tucker 1966).

Three-way Nonsymmetric Correspondence Analysis (NSCA3) as three-way Correspondence Analysis (Carlier & Kroonenberg 1994) allows a detailed analysis of the deviations from independence. It is based on the decomposition of a nonsymmetric association index which allows to know how the second and third variables predict the first one. The more different the index value from zero is the higher the the predictivity power is. in combination with Lancaster's (1951) additive decomposition of interactions among the variables, in the space $\Re^{I \times J \times K}$, it will be shown the decomposition of the tridimensional measure of predicability on which the analysis is based (τ, Marcotorchino 1984). We will also pay attention to Gabriel's (1971) biplot technique which allows to keep the initial non-symmetry of the method in the graphical representation (Lombardo & Kroonenberg 1993).

2 Notation

Let Y, X and Z be respectively the responce and the predictor variables observed on N objects, and I,J,K the categories numbers. Let $\Re^{\mathsf{I} \times \mathsf{J} \times \mathsf{K}}$ be the space of real functions depending on i, j e k (for $i = 1...I, j = 1...J, k = 1...K$). Let $\mathbf{N}$ be the contingency table of order $\mathsf{I} \times \mathsf{J} \times \mathsf{K}$ whose general term is $\{n_{ijk}\}$, and $\mathbf{P}$ the relative frequency table associated to $\mathbf{N}$ whose general term is $\{p_{ijk}\}$, such that $\sum_i \sum_j \sum_k p_{ij} = 1$, and let $p_{i..} = \sum_j \sum_k p_{ijk}$, $p_{.j.} = \sum_i \sum_k p_{ijk}$, $p_{..k} = \sum_i \sum_j p_{ijk}$, $p_{ij.} = \sum_k p_{ijk}$, $p_{i.k} = \sum_j p_{ijk}$, $p_{.jk} = \sum_i p_{ijk}$ be the marginal frequencies (a dot indicates the sum with respect to an index i, j or k). Let I, the identity matrix of order $\mathsf{I} \times \mathsf{I}$, be the metric of the space $\Re^{\mathsf{I}}$, and let D_{J}, D_{K} be the diagonal metrics of $\Re^{\mathsf{J}}$ and $\Re^{\mathsf{K}}$, whose general terms are, respectevily, $\{p_{.j.}\}$ and $\{p_{..k}\}$. Let **a** and **e** be vectors (both of order $I \times 1$)in $\Re^{\mathsf{I}}$, **b** and **f** (both of order $J \times 1$) in $\Re^{\mathsf{J}}$, and **c** and **g** (both of order $K \times 1$) in $\Re^{\mathsf{K}}$, the inner products between two vectors, respectively, in the space $\Re^{\mathsf{I}}$, $\Re^{\mathsf{J}}$ and $\Re^{\mathsf{K}}$ are so defined: $< a, e >_{\mathsf{I}} = \sum_i a_i e_i$; $< b, f >_{\mathsf{D_J}} = \sum_j p_{.j.} b_j f_j$; $< c, g >_{\mathsf{D_K}} = \sum_k p_{..k} c_k g_k$. Given the definition of the inner products in $\Re^{\mathsf{I}}, \Re^{\mathsf{J}}, \Re^{\mathsf{K}}$, an usual extension (Franc 1992) of the inner product definition in the space $\Re^{\mathsf{I \times J \times K}}$ between the three-way matrices $\mathbf{X}$=$\{x_{ijk}\}$ and $\mathbf{Y}$=$\{y_{ijk}\}$ can be written as:

$$< \mathbf{X}, \mathbf{Y} >= \sum_{ijk} p_{.j.} p_{..k} x_{ijk} y_{ijk}$$

and the quadratic norm as

$$\| \mathbf{X} \|^2 = \sum_{ijk} p_{.j.} p_{..k} x_{ijk}^2.$$

Defining $\mathbf{X}$ and $\mathbf{Y}$ as three-way tables whose (i, j, k) element are, respectively, equal to $\{x_{ijk} = a_{ip} b_{jq} c_{kr}\}$ and $\{y_{ijk} = e_{ip} f_{jq} g_{kr}\}$, in the same way, the inner product between $\mathbf{X}$ and $\mathbf{Y}$ in $\Re^{I \times J \times K}$ is, also, so defined: $< \mathbf{X}, \mathbf{Y} > =< a, e >_I < b, f >_{D_J} < c, g >_{D_K}$.

3 The measure of dependence

In order to study the dependence among variables it could be preferred a measure which is explicitly sensitive to the asymmetry of the situation. We will concentrate on a measure proposed by Marcotorchino (1984) called generally τ, which represents the three-way generalization of the predictability measure proposed by Goodman & Kruskal (1954). One of the aims of predicting the value of a variable from the others is the reduction of uncertainty about the value of the responce variable from the knowledge of the predictor variables. It seems, therefore, sensible to use a measure like τ which measures the reduction of uncertainty, i.e.

$$\tau = \frac{\sum_{ijk} (p_{ijk})^2 / p_{.j.} p_{..k} - \sum_i p_{i..}^2}{1 - \sum_i p_{i..}^2}. \quad (1)$$

In particular τ indicates the proportional difference between predicting an object's category on the basis of the marginal distribution of the responce variable Y and on the basis of the conditional distribution of Y given that we know that an object is in a particulary category of the predictors X and Z. The total proportion of correct predictions, solely based on the marginal distribution of Y is $\sum_i p_{i..}^2$, and the total proportion of incorrect predictions is $1 - \sum_i p_{i..}^2$. Similarly the total proportion of correct predictions, given the knowledge of X and Z, is close to $\sum_{ijk} p_{ijk}^2 / p_{.jk}$ (Gray & Williams 1981). Therefore the total proportion of incorrect predictions is $1 - \sum_{ijk} p_{ijk}^2 / p_{.jk}$. When X and Z carry no information about Y, then there should be no relative decrease in the errors of prediction, τ is zero. The more information is given by X and Z about Y, the higher the τ value is. If it is verified that $\sum_i p_{ijk} = p_{.j.}p_{..k}$ then the expression (1) is equivalent to the Gray & Williams'τ index.

Observe that τ is based on the deviations from the three-way independence model, so it contains the information on all two-way and three-way interactions. The measure which will be decomposed is the absolute and not the relative decrease in proportion of incorrect predictions i.e. the τ numerator. This has no real influence on the results of the decomposition. The τ numerator can be expressed in the following way :

$$\begin{aligned} N_\tau &= \sum_{ijk} \frac{p_{ijk}^2}{p_{.j.}p_{..k}} - \sum_i p_{i..}^2 \\ &= \sum_{ijk} p_{.j.}p_{..k} \left(\frac{p_{ijk}}{p_{.j.}p_{..k}} - p_{i..} \right)^2 \end{aligned}$$

Let $\|\mathbf{\Pi}\|^2$ be the quadratic norm of the three-way matrix $\mathbf{\Pi}$ whose general term is $\pi_{ijk} = p_{ijk}/p_{.j.}p_{..k} - p_{i..}$, then N_τ can also be written as:

$$N_\tau = \sum_{ijk} p_{.j.}p_{..k} (\pi_{ijk})^2 = \|\mathbf{\Pi}\|^2 \tag{2}$$

The Inertia N_τ measures the absolute reduction in uncertainty when we predict the category i given that j and k have occurred.

4 Nonsymmetric Correspondence Analysis for three-way contingency tables

Nonsymmetric Correspondence Analisys for three-way tables is based on the decomposition of the tridimensional nonsymmetric Marcotorchino's index. In order to find an appropriate model for the measure expressed in formula (2) we look at a particular type of Singular Value Decomposition (SVD) generalizations for three-way arrays. In particular, among several possible generalizations, the Tucker3 model (three-mode factor analysis model, Tucker 1966) will be employed. With this method the global measure of dependence is expressed as

$$\pi_{ijk} = \sum_{p=1}^{P}\sum_{q=1}^{Q}\sum_{r=1}^{R} g_{pqr}a_{ip}b_{jq}c_{kr} + e_{ijk}$$

where P,Q and R ($\mathsf{P} \leq \mathsf{I}; \mathsf{Q} \leq \mathsf{J}; \mathsf{K} \leq \mathsf{R}$) represent the fixed number of the component matrices A, B and C, respectively. It will be searched for an approximation $\hat{\mathbf{\Pi}} = \{\hat{\pi}_{ijk}\}$ which minimizes

$$Min \sum_{ijk} p_{.j.}p_{..k}(\pi_{ijk} - \hat{\pi}_{ijk})^2$$

where $\hat{\pi}_{ijk} = \sum_p^P \sum_q^Q \sum_r^R g_{pqr}a_{ip}b_{jq}c_{kr}$ and the parameters a_{ip}, b_{jq}, c_{kr} and g_{pqr} are those which optimize the function

$$Min\{\sum_{ijk} p_{.j.}p_{..k}e_{ijk}^2\}$$

The set of vectors $\{\mathbf{a}_p\}; \{\mathbf{b}_q\}; \{\mathbf{c}_r\}$ are taken to be orthonormal in their respective spaces $\Re^{\mathsf{I}}$, $\Re^{\mathsf{J}}$, $\Re^{\mathsf{K}}$, without restrictions of generality, because it can be computed the same infinity number of solutions with or without the orthonormality constraintes. The term g_{pqr} represents the general element of the core matrix and can be interpreted as the generalization of a "singular value" and the term e_{ijk} represents the error of approximation. For an extensive discussion of the Tucker3 model see for instance Kroonenberg (1983). The Inertia N_τ can be split into an explained part and a residual part.

$$\|\mathbf{\Pi}\|^2 = \|\hat{\mathbf{\Pi}}\|^2 + \|\mathbf{e}\|^2$$

Note that three-way nonsymmetric correspondence analysis uses a *weighted* alternating least-squares algorithm (Tuckals3; Kroonenberg 1983, Lombardo 1994).

Additional property. As consequence of the chosen model, the orthogonality of the vectors $\mathbf{a}_p = \{a_{ip}\}, \mathbf{b}_q = \{b_{jq}\}, \mathbf{c}_r = \{c_{kr}\}$ implies the orthogonality of the array $a_p \otimes b_q \otimes c_r$, defined as the tensorial product of the three vectors, whose general term is $\{a_{ip}b_{jq}c_{kr}\}$ (for $p = 1...\mathsf{P}, q = 1...\mathsf{Q}, r = 1...\mathsf{R}$) in $\Re^{\mathsf{I}\times\mathsf{J}\times\mathsf{K}}$. Denoting $\hat{\mathbf{\Pi}} = \sum_{ijk} g_{pqr}\mathbf{a}_p \otimes \mathbf{b}_q \otimes \mathbf{c}_r$ we have that the quadratic norm of $\hat{\mathbf{\Pi}}$ can be written as:

$$\|\hat{\mathbf{\Pi}}\|^2 = \sum_{pqr} g_{pqr}^2$$

it shows that the explained part of $\mathbf{\Pi}$ that is the tau index numerator (N_τ) can be expressed in terms of each element of the core matrix.

5 Measures of partial dependence

To study the two-way interactions (the first order interactions) and the three-way interaction (the second order interaction) among the variables, the matrix $\mathbf{\Pi}$ can be decomposed orthogonally in the euclidean space $\Re^{\mathsf{I}\times\mathsf{J}\times\mathsf{K}}$.

The space $\Re^{\mathsf{I}\times\mathsf{J}\times\mathsf{K}}$ (vectorial space of real functions) can be decomposed in subspaces whose direct sum is an orthogonal decomposition (see also Carlier & Kroonenberg 1994):

$$\Re^{\mathsf{I}\times\mathsf{J}\times\mathsf{K}} = C_{\mathsf{o}} \oplus C_{\mathsf{I}} \oplus C_{\mathsf{J}} \oplus C_{\mathsf{K}} \oplus C_{\mathsf{IJ}} \oplus C_{\mathsf{IK}} \oplus C_{\mathsf{JK}} \oplus C_{\mathsf{IJK}}$$

where C_{o} represents the subspace of real functions which doen not depend on i, j or k, C_{I} is the subspace of real functions which depends on i (per $i = 1...I$) etc.. In general let $\mathbf{X}$ be the three-way array of $\Re^{\mathsf{I}\times\mathsf{J}\times\mathsf{K}}$, it is possible to define the following orthogonal additive decomposition

$$x_{ijk} = a + b_i + c_j + d_k + e_{ij} + f_{ik} + g_{jk} + h_{ijk} \tag{3}$$

where $a, b_i, c_j, d_k, e_{ij}, f_{ik}, g_{jk}, h_{ijk}$ represent, respectively, the orthogonal projections of $\mathbf{X}$ onto the subspaces C_{o}, C_{I}, C_{J} ,C_{K}, C_{IJ}, C_{IK}, C_{JK}, C_{IJK}.
The anova-like decomposition of $\mathbf{X}$ is the following

a=$x_{...}$

$b_i = x_{i..} - x_{...}$

$c_j = x_{.j.} - x_{...}$

$d_k = x_{..k} - x_{...}$

$e_{ij} = x_{ij.} - x_{i..} - x_{.j.} + x_{...}$

$f_{ik} = x_{i.k} - x_{i..} - x_{..k} + x_{...}$

$g_{jk} = x_{.jk} - x_{.j.} - x_{..k} + x_{...}$

$h_{ijk} = x_{ijk} - x_{ij.} - x_{i.k} - x_{.jk} + x_{i..} + x_{.j.} + x_{..k} - x_{...}$

In the same way we look at the orthogonal additive decomposition of $\mathbf{\Pi}$ in $\Re^{\mathsf{I}\times\mathsf{J}\times\mathsf{K}}$. It can be proved, (for some technical details see Lombardo 1994), that the orthogonal projection of $\mathbf{\Pi}$ onto the subspaces $C_{\mathsf{o}}, C_{\mathsf{I}}, C_{\mathsf{J}}, C_{\mathsf{K}}$ are null ($\pi_{ijk} = e_{ij}+f_{ik}+g_{jk}+h_{ijk}$) so that the unique orthogonal decomposition of $\mathbf{\Pi}$ is equal to:

$$\left(\frac{p_{ijk}}{p_{.j.}p_{..k}} - p_{i..}\right) = \left(\frac{p_{ij.}}{p_{.j.}} - p_{i..}\right) + \left(\frac{p_{i.k}}{p_{..k}} - p_{i..}\right) \tag{4}$$

$$+ \left(\frac{p_{.jk}}{p_{.j.}p_{..k}} - 1\right) + \left(\frac{p_{ijk} -_{\alpha} p_{ijk}}{p_{.j.}p_{..k}}\right) \tag{5}$$

where the element $_{\alpha}p_{ijk}$ is implicitely defined in the equation (5). Each term of the decomposition defines a three-way matrix which depends only on some of the indences i, j and k. We observe that each slice of the three-way matrices defined by the two-way interaction terms is a Torgerson's matrix (which are double-centered and with at least a singular value equal to zero).

Equation (5) may also be written in terms of matrices:

$$\|\mathbf{\Pi}\| = \|\mathbf{\Pi}_{[\mathbf{IJ}]}\| + \|\mathbf{\Pi}_{[\mathbf{IK}]}\| + \|\mathbf{\Pi}_{[\mathbf{JK}]}\| + \|\mathbf{\Pi}_{[\mathbf{IJK}]}\|$$

The additive partitioning of the quadratic norm of $\mathbf{\Pi}$ can be written as

$$\|\mathbf{\Pi}\|^2 = \|\mathbf{\Pi}_{[\mathbf{IJ}]}\|^2 + \|\mathbf{\Pi}_{[\mathbf{IK}]}\|^2 + \|\mathbf{\Pi}_{[\mathbf{JK}]}\|^2 + \|\mathbf{\Pi}_{[\mathbf{IJK}]}\|^2$$

This equation may also be expressed in the following way:

$$N_{\tau} = \sum_{ijk} p_{.j.}p_{..k}\left(\frac{p_{ijk}}{p_{.j.}p_{..k}} - p_{i..}\right)^2 = \sum_{ij} p_{.j.}\left(\frac{p_{ij.}}{p_{.j.}} - p_{i..}\right)^2 \tag{6}$$

$$+ \sum_{ik} p_{..k}\left(\frac{p_{i.k}}{p_{..k}} - p_{i..}\right)^2 \tag{7}$$

$$+ \sum_{jk} \frac{1}{\mathsf{I}} p_{.j.}p_{..k}\left(\frac{p_{.jk}}{p_{.j.}p_{..k}} - 1\right)^2 \tag{8}$$

$$+ \sum_{ijk} p_{.j.}p_{..k}\left(\frac{p_{ijk} -_{\alpha} p_{ijk}}{p_{.j.}p_{..k}}\right)^2 \tag{9}$$

The τ numerator (inertia) is so decomposed in partial indeces. The terms (6) and (7) represent two measures of predicability (τ, Goodman & Kruskal 1954) between the I categories of the responce variable Y and the J categories of the predictor X (they are the elements approximated by two-way Nonsymmetric Correspondence Analysis, Lauro & D'Ambra 1984; D'Ambra & Lauro 1989, 1992). The term (8) represents the symmetric measure of the interdependence between the J categories of X and the K categories of Z (χ^2/N except for the costant 1/I). Further the last term represents a measure of the three-way interaction among the variables. The most important property of this decomposition is that it leads to an additive partitioning of the interactions in a three-way array. In case of the symmetric measure $\Phi^2 = \chi^2/N$ (Pearson 1900) the additive decomposition was introduced by Lancaster (1951, 1960).

Special case: Nonsymmetric Correspondence Analysis Multiple (D'Ambra & Lauro 1989). In case the two-way interaction term $\mathbf{\Pi_{[JK]}}$ turns out to be very near to zero ($p_{.jk} = p_{.j.}p_{..k}$), it can be verified that Marcotorchino's τ (2) is coincident to the τ index proposed by Gray & Williams (1981) which is on the basis of Nonsymmetric Correspondence Analysis Multiple (D'Ambra & Lauro, 1989). It means that τ, obtained from the three-way table, is equal to the τ of the two-way table obtained by coding J and K interactively. The approximation of $\mathbf{\Pi}$ can be obtained by Multiple NSCA on the table with rows indexed by i and columns by (j,k). For an example see Lombardo (1994).

6 Modeling of partial dependence

The orthogonal additive decomposition of $\mathbf{\Pi}$ and the Tucker3 model allow to study the partial terms of the interaction. Note that the interaction terms can be obtained as partial means of π_{ijk} respect to one of the indeces i, j, k. For example the general term of $\mathbf{\Pi_{[IJ]}}$ is

$$\pi_{[ij.]} = \sum_k p_{..k}\left(\frac{p_{ijk}}{p_{.j.}p_{..k}} - p_{i..}\right) = \frac{p_{ij.}}{p_{.j.}} - p_{i..} \tag{10}$$

The sub-models for such part can be obtained in the following manner:

$$\frac{p_{ij.}}{p_{.j.}} - p_{i..} = \sum_{pqr} g_{pqr}a_{ip}b_{jq}\left(\sum_k p_{..k}c_{kr}\right)$$

Note that it could be possible to analyse the dependence among the variables after removing the effect due to some margin matrices. For example subtracting the matrix of margin $\mathbf{\Pi_{[IJ]}}$ from $\mathbf{\Pi}$ we compute the part of Inertia explained by the margins $I \times K$, $J \times K$ and $I \times J \times K$:

$$\pi - \pi_{ij.} = \pi_{i.k} + \pi_{.jk} + \pi_{ijk} \tag{11}$$

$$\sum_{pqr} g_{pqr}a_{ip}b_{jq}c_{kr} - \sum_{pqr} g_{pqr}a_{ip}b_{jq}c_{.r} = \sum_{pqr} g_{pqr}a_{ip}b_{.q}c_{kr} \tag{12}$$

$$+ \sum_{pqr} g_{pqr}a_{+p}b_{jq}c_{kr} \tag{13}$$

$$+\quad \sum_{pqr} g_{pqr}(a_{ip}-a_{.p})(b_{jq}-b_{.q})(c_{kr}-c_{.r}) \quad (14)$$

It is clear that in case of approximation of $\mathbf{\Pi}$ a different expression of the interaction partial terms can be computed because the projections of $\mathbf{\Pi}$ onto the subspaces $C_o, C_{\mathsf{I}}, C_{\mathsf{J}}, C_{\mathsf{K}}$ will be only approximatively equal to zero (Carlier & Kroonenberg 1994; Lombardo 1994).

7 Graphical representation of the dependence

In order to maintain the non-symmetry of the analysis in the graphical representation the biplot technique will be proposed. The *Interactive Biplot* (Carlier & Kroonenberg 1994) and the *Joint Biplot* (see Kroonenberg, 1983, p. 164), to display the dependence among the variables, will be considered. First it is useful to remind some definitions in relation with the standard biplots and the calibrated biplots.

Standard Biplots: A matrix X ($\mathsf{I}\times\mathsf{J}$) of rank $\mathsf{S_o}=2$ is decomposed into a product of two matrices: $\hat{X}=YZ'$ where $Y=\{y_{is}\}$ is $\mathsf{I}\times\mathsf{S_o}$ and $Z=\{z_{js}\}$ is $\mathsf{J}\times\mathsf{S_o}$ (Gabriel 1971). Using the rank-two approximation of $\hat{X}$ each element of this matrix can be written as $\hat{x_{ij}}=y_{i1}z_{j1}+y_{i2}z_{j2}$. To display the matrix $\hat{X}$ each row is represented as a point Y_i (row markers) with coordinates (y_{i1},y_{i2}) and each column as a point Z_j (column markers) with coordinates (z_{j1},z_{j2}) in a two-dimensional figure with origin O. Let Y_i'' be the orthogonal projection of Y_i on the arrow $\vec{OZ_j}$ and α_{ij} the angle between the vectors $\vec{OY_i}$ and $\vec{OZ_j}$ then the approximated element of X can be written as

$$\hat{x_{ij}}=\|\vec{OZ_j}\|\|\vec{OY_i}\|cos(\alpha_{ij})=\pm\|\vec{OZ_j}\|\|\vec{OY_i''}\|$$

The value $\hat{x_{ij}}$ is proportional to the length $\vec{OY_i''}$ is positive if the angle between $\vec{OZ_j}$ and $\vec{OY_i''}$ is acute, negative in the case of an obtuse angle and null in the case of an orthogonal angle. The three most important factorisations of $\hat{X}$, based on the Singular Value Decomposition, are the following

$$\text{row isometric factorisation: } \hat{x}_{ij}=\sum_s(\lambda_s a_{is})b_{js}$$

$$\text{column isometric factorisation: } \hat{x}_{ij}=\sum_s a_{is}(\lambda_s b_{js})$$

$$\text{symmetric factorisation: } \hat{x}_{ij}=\sum_s(\sqrt{\lambda_s}a_{is})(\sqrt{\lambda_s}b_{js})$$

As soon as NSCA aims to portray the nonsymmetric relationship between the row and column categories, that means how the columns predict the rows, we portray this in a figure by means a column isometric biplot. Using the biplot representation in Nonsymmetric Correspondence Analysis for two-way table we have a correct representation of the row-column relationship combined with a column-metric preserving (column isometric biplot, Lombardo & Kroonenberg 1993; Kroonenberg & Lombardo 1994). The markers of the columns will be associated with axis or arrows and the row markers with points. In summary the following rules will be used in interpreting the column isometric biplot:

- only distances for column markers can be properly judged from the biplot
- the larger the inner product between a row and a column marker is the larger the predictive power of that column for that row category is.
- the origin represents the row margin, predictors with large distances from the origin show good reductions in error of prediction.

The advantage of the biplot graph is connected with the clear representation of the relationship between the row and column markers differently from the classical representations.

Calibrated Biplots (Gabriel & Odoroff 1990, Greenacre 1993, Carlier & Kroonenberg 1994). In such kind of biplots the approximation of x_{ij} can be directly read from the graph. As the value of $\hat{x}_{ij}$ depends only on the orthogonal projection Y_i'' on the arrow $\vec{OZ}_j$ and is proportional to the length $\|\vec{OY}_i''\|$, it is possible to mark the axis $\vec{OZ}_j$ and read, directly on it, the approximated values $\hat{x}_{ij}$ projecting the points Y_i orthogonally onto $\vec{OZ}_j$. The calibration of a biplot arrow is computed inverting the length of each arrow in such way we compute the interval length of tick marks on the axes (the scale is positionated on the arrow which will be called axes; the j^{th} axes could be decentered with respect to the mean value so the origin indicates the true mean value for the j^{th} variable).

Interactive Biplot (Carlier & Kroonenberg 1994) The *Interactive Plot* is based on the following decomposition

$$\hat{\pi}_{ijk} = \sum_p a_{ip}[\sum_{qr} g_{pqr} b_{jq} c_{kr}] \quad (15)$$

$$= \sum_p a_{ip} d_{(jk)p} \quad (16)$$

where $d_{(jk)p}$ denotes the term in square brackets (15), it represents the inner product between the predictor categories i, j ($\forall i \epsilon \mathsf{I}$ and $\forall j \epsilon \mathsf{J}$). The matrix $\hat{\mathbf{\Pi}}$ is approximated by the linear combination of the matrices $D_p = (d_{(jk)p})$ (for $p = 1...\mathsf{P}$) with coefficients a_{ip} (for $i = 1...\mathsf{I}$). Each pair of predictor categories (j, k) will be represented by a single point, the second and third variable are coded interactively. The interactive biplots are very useful if the categories number ($J \times K$) is not too large, or when one of the modes is ordered (for example the time).

Joint Biplots (Kroonenberg 1983, Carlier & Kroonenberg 1994). The *Joint biplots and plots* are based on the same equation (15). For each index p a Singular Value Decomposition of the matrix D_p is performed and the resulting coordinates are represented in the graph (Kroonenberg 1983, Carlier & Kroonenberg 1994). If an approximation of rank two is chosen there will be P biplots to observe. Note that to represent the interactions among the I categories of the responce variable and the J (or K) categories of the predictor the following equation is proposed

$$\hat{\pi}_{ijk} = \sum_r c_{kr}[\sum_{pq} g_{pqr} a_{ip} b_{jq}] \quad (17)$$

$$= \sum_r c_{kr} d_{(ij)r} \quad (18)$$

where $d_{(ij)r}$ represents the general term of the matrix D_r which contains the inner products among the I and the J categories. In this case $\hat{\mathbf{\Pi}}$ is approximated by the linear combination of the matrices $D_r = (d_{(ij)r})$ (for $r = 1...\mathsf{R}$) with coefficients c_{kr} (for $k = 1...\mathsf{K}$). For each index r the SVD of D_r is performed. The coordinates will be graphed in a biplot which maintains the distances among the J categories (column isometric biplot, Lombardo & Kroonenberg 1993).

8 An application: Study type choice in function of parents professional occupations and sex

In this section it will be discussed a sociological example, we will study the relationship between study types chosen by french students and occupational categories of their parents. Our aim is to point out the dependence between the responce variable *studies type* and the predictors *parents occupations* and *sex*. We consider thirty-four occupational categories of parents: *farmer* **farm**, *farm labourer* **lfar**, *manifacturer* **indu**, *self employed craftsman* **craf**, *skipper* **skip**, *middle and bigger trader* **mbtr**, *shopkeeper* **shop**, *professional people* **ppro**, *literary and scientifique occupation* **lsoc**, *engeneer* **enge**, *upper administrative manager* **uman**, *high school teacher* **prof**, *salaried and medical occupation* **medo**, *social medical profession* **medp**, *technician* **tech**, *middle administrative manager* **mman**, *teacher* **teach**, *miscellaneous intellectual occupation* **intp**, *study hall teacher* **stea**, *office clerk* **ocle**, *business employed* **ebus**, *foreman* **enmt**, *skilled workman* **skil**, *semi-skilled workman* **sski**, *miner* **mine**, *fisherman* **fish**, *unskilled workman* **unsk**, *domestic servants* **dome**, *artist* **arts**, *clergy man* **clrg**, *army and police* **arpo**, *other categories* **othe**, *unemployed* **unem** *and non available* **nona**.

I need the data source

The type of studies considered are: *law* **Law**, *exact sciences* **Sciences**, *humanities* **Human**, *medicine* **Medic**, *odontology* **Odont**, *pharmacology* **Pharmac**, *technology* **Techn**. The data, (table 1), are in relation with the sex *males* and *females* of the french students

The τ decomposition. In table 2 we read the values of tau numerator decomposition. The tridimensional index value is ($\tau = 0.054$). Given the logical dependence between study typies and parents'occupations we decide to investigate further the data.
We see that the *study type* $\times$ *sex* interaction is the largest one (67%) while the *study type* $\times$ *occupation* interaction is of primary interest. The change in the study type choice between males and females students, suggests that males choose differently from females students the study type to undertake.

Graphical representation. In order to assess how many components are adequate for any analysis we investigate how much each combination of components contributes to the overall fit. In this sense the square core elements could be interpreted as generalizations of the singular values. We first looked at the core matrix with P=7, Q=34, R=2. The seven components of the first mode are indicated, respectively, with $p_1 ... p_7$; the thirty-four components of the second mode are $q_1 ... q_{34}$; the two components of the third are r_1 and r_2. We see that it is not useful to choose a model with $\mathsf{P} > 3; \mathsf{Q} > 4$ because the contributions to the inertia of the component combinations, when $P > 3$ and $Q > 4$ are almost zero. For this reason we redid the analysis with these number of components $\mathsf{P} = 3, \mathsf{Q} = 4, \mathsf{R} = 2$ leading to a fit of 98% (table 3).

Joint Biplot. For this example it is not very useful to represent the results by the Interactive Biplot because the professional occupation category number is very large so that it seems a good choice to use *Joint Biplots* for visualizing the results of the decomposition.

In reality I have not seen the Interactive Biplot! I will do it this afternoon

We will now discuss some principles important for the interpretation. In order to explain the results we look at the matrices D_r that in linear combination with the coefficients c_{kr} allow the representation of $\mathbf{\Pi}$. The $d_{i(jr)}$ $(r = 1, 2)$ elements can be interepreted as the inner products between the marker of the study type category i and the occupational category j in the joint biplot based upon $D_r (= d_{i(jr)},\ for\ i = 1..7,\ j = 1..34)$. The interactions among the *study type* categories and the *occupational* categories are visualized in $r = 2$ graphs (equation 18). The sex effect is in relation with the c_{kr} coefficients. Further we see that the weigthed mean of $\mathbf{\Pi}$ with respect to the index k represents the partial interaction matrix $\Pi_{[IJ]}$ (formula 10).

- The coefficients c_{k2} are approximately equal to one ($c_{12} = 0.9$, $c_{22} = 1.1$), the mean value is $c_{.2} = 0.1$. As a first approximation we can say that the slice

Table 1: *Study types in relation with the parents'occupational status and sex*

	Males											
	farm	**lfar**	**indu**	**craf**	**skip**	**mbtr**	**shop**	**ppro**	**lsoc**	**enge**	**uman**	**prof**
Law	3649	362	2123	2189	41	1837	3347	6858	269	3042	10106	1521
Sciences	5032	699	823	2518	27	785	2802	2220	156	2816	4824	1946
Human	3180	479	523	1679	20	649	2195	1786	85	1050	3676	1626
Medic	1410	101	910	1074	19	926	1869	5404	73	2168	4462	1068
Odont	103	9	89	117	0	108	255	470	5	188	384	109
Pharmac	295	23	152	166	0	208	361	1043	21	284	608	159
Techn	968	173	180	546	6	190	556	174	6	219	461	100
	medo	**medp**	**tech**	**mman**	**teach**	**intp**	**stea**	**ocle**	**ebus**	**fore**	**skil**	**sski**
Law	661	297	1392	5498	1106	674	196	4195	1779	1204	2655	1715
Sciences	288	247	1814	4676	2498	385	76	3958	1245	1547	3338	2519
Human	238	197	1169	3530	1964	318	66	3414	1272	1080	2734	1796
Medic	695	208	878	2591	1086	317	45	1606	872	588	836	745
Odont	42	18	66	269	136	25	9	151	101	38	65	53
Pharmac	48	23	111	333	162	30	5	195	101	47	90	70
Techn	28	28	313	368	197	78	46	518	223	250	680	632
	mine	**fish**	**unsk**	**dome**	**arts**	**Clrg**	**arpo**	**othe**	**unem**	**nona**		
Law	278	113	571	450	155	47	2024	3422	2522	5142		
Sciences	610	180	964	447	65	71	1586	4003	1603	3333		
Human	489	140	799	470	73	84	1170	2493	1524	3227		
Medic	204	56	231	118	34	84	671	1622	918	877		
Odont	8	2	12	15	1	3	49	187	61	86		
Pharmac	9	7	24	15	3	1	52	236	95	452		
Techn	115	26	158	115	13	3	139	423	277	229		
	Females											
	farm	**lfar**	**indu**	**craf**	**skip**	**mbtr**	**shop**	**ppro**	**lsoc**	**enge**	**uman**	**prof**
Law	1906	210	1137	1188	18	834	1633	2958	104	1690	4191	715
Sciences	3264	378	396	1361	15	517	1469	1120	74	1650	2375	1043
Human	5849	700	1825	3611	36	1873	4483	4780	185	3386	8364	3095
Medic	764	66	493	525	10	459	750	2219	38	1139	1962	608
Odont	52	3	30	41	1	31	76	153	2	61	98	26
Pharmac	566	37	344	277	2	310	499	1365	26	597	918	231
Techn	274	46	52	90	0	50	108	64	0	100	137	43
	medo	**medp**	**tech**	**mman**	**teach**	**intp**	**stea**	**ocle**	**ebus**	**fore**	**skil**	**sski**
Law	345	131	810	2636	542	365	92	1793	794	625	1370	943
Sciences	141	113	996	2282	1272	174	24	1811	584	678	1542	1222
Human	667	351	2624	6896	3429	641	144	5651	2227	1963	4239	2919
Medic	295	67	439	1148	519	144	16	577	379	246	331	313
Odont	14	8	22	71	31	6	2	24	26	7	6	13
Pharmac	91	29	161	439	268	39	17	215	129	63	112	107
Techn	12	28	67	149	60	15	17	89	45	42	81	115
	mine	**fish**	**unsk**	**dome**	**arts**	**Clrg**	**arpo**	**othe**	**unem**	**nona**		
Law	136	60	284	210	44	18	799	1880	1221	2787		
Sciences	197	87	428	170	17	36	634	2008	767	1420		
Human	557	220	946	590	114	105	2064	4799	2515	4603		
Medic	66	25	81	58	21	37	244	648	395	304		
Odont	3	2	2	4	1	0	17	47	12	22		
Pharmac	10	10	28	16	4	3	98	313	66	532		
Techn	7	4	40	25	1	5	41	77	47	57		

Table 2: τ *numerator values and total and partial indeces*

	N$_\tau$	τ	$\approx \chi^2$	**% of total**
Two-way interactions				
Study x Occup.	.012	0.016		29.4%
Study x Sex	.028	0.036		66.9 %
Occup. x Sex			0.000	0.7 %
Three-way interaction			0.001	3.0%
Tridimensional τ	.041	0.054		100%

D_2 does not explain any sex effect, but it describes the general dependence between the type of studies chosen by the students and the parents'occupation category.

- The coefficients $c_{k1}(= 1.0, -1.0)$ have a mean equal to one ($c_{.1} = 1$). The corresponding biplot portrays what is different between males and females students.

Further as soon as $c_{.1}$ is close to zero and $c_{.2}$ is one, we can say that the matrix D_2 is approximately equal to the matrix $\Pi_{[IJ]}(= \pi_{[ij.]})$:

$$\pi_{[ij.]} = \sum_k p_{..k}\pi_{ijk} = c_{.1}d_{ij1} + c_{.2}d_{ij2}.$$

The graph interpretation should be taken into account the coefficients c_{kr} and the inner products $d_{i(jr)}$, $\forall i, j, r$.
In order to understand what is different between males and females students we look the biplot representation of slice D_1. Taking in mind the c_{kr} coefficient sign, we interpret positevely the interactions for the males and negatively for females. The opposition on the first axis of D_1 graph shows the preference of females students

Table 3: *Contributions to the Inertia*

	r1							
	q1	**q2**	**q3**	**q4**				
p1	**65**	0	0	0	1	9	0	1
p2	1	2	0	0	0	12	0	1
p3	0	0	1	0	0	0	4	0

Table 4: *Scalar product matrices D_r*

						D_1						
	farm	**lfar**	**indu**	**craf**	**skip**	**mbtr**	**shop**	**ppro**	**lsoc**	**enge**	**uman**	**prof**
Law	29.2	11.2	**102.8**	47.8	64.9	86.8	56.5	81.5	**91.6**	54.1	90.2	36.7
Sciences	**65.9**	**74.7**	21.3	61.5	32.2	26.3	44.3	-8.4	38.1	37.5	27.0	57.0
Human	-127.9	-105.2	-171.7	-153.9	-133.6	-169.9	-149.8	-155.0	-152.7	-159.8	-159.7	-149.9
Medic	21.4	9.3	41.5	31.9	29.6	46.9	37.4	68.1	21.0	52.0	36.5	40.2
Odont	3.4	2.2	5.2	4.5	4.0	5.7	4.8	7.1	3.5	6.1	4.7	5.1
Pharmac	-12.3	-14.0	-6.7	-11.5	-7.4	-5.9	-8.2	3.6	-11.5	-5.1	-7.5	-8.9
Techn	16.3	17.6	8.6	16.1	9.8	9.4	12.5	1.8	11.6	11.2	9.4	14.9
	medo	**medp**	**tech**	**mman**	**teach**	**intp**	**stea**	**ocle**	**ebus**	**fore**	**skil**	**sski**
Law	70.0	51.4	42.5	60.9	14.0	77.1	**107.1**	54.9	62.8	43.4	39.4	29.8
Sciences	-0.1	35.6	62.6	48.3	**70.6**	27.6	30.3	57.3	39.7	65.1	**66.8**	**66.8**
Human	-165.2	-141.6	-158.2	-148.8	-143.5	-141.3	-151.0	-139.7	-141.5	-146.0	-125.3	-126.2
Medic	76.6	41.9	37.9	30.4	39.5	30.6	15.7	20.2	30.8	26.1	11.9	19.1
Odont	7.9	5.0	5.0	4.3	5.1	4.1	3.0	3.4	4.2	4.0	2.6	3.2
Pharmac	3.8	-5.7	-10.7	-10.1	-10.0	-7.3	-11.7	-12.6	-8.6	-12.6	-14.2	-12.7
Techn	3.6	10.5	16.3	13.4	17.4	9.1	10.1	15.0	11.4	16.6	16.6	16.5
	mine	**fish**	**unsk**	**dome**	**arts**	**Clrg**	**arpo**	**othe**	**unem**	**nona**		
Law	2.1	25.7	11.3	41.1	**105.3**	-1.4	72.4	46.0	70.5	73.6		
Sciences	**64.1**	**67.8**	63.8	51.1	33.7	23.2	52.7	59.0	36.8	38.1		
Human	-97.0	-126.0	-90.4	-99.4	-149.4	-103.3	-149.9	-141.9	-128.7	-105.4		
Medic	18.1	20.8	7.2	4.4	13.0	59.0	19.5	26.3	18.3	-2.2		
Odont	2.8	3.4	1.9	1.7	2.8	6.0	3.4	3.9	3.0	1.1		
Pharmac	-10.3	-12.4	-12.2	-12.2	-12.5	2.4	-13.1	-11.6	-10.0	-12.8		
Techn	15.1	16.7	15.1	12.9	10.8	6.6	14.5	15.3	10.7	10.6		
						D_2						
	farm	**lfar**	**indu**	**craf**	**skip**	**mbtr**	**shop**	**ppro**	**lsoc**	**enge**	**uman**	**prof**
Law	-57.6	-83.5	87.6	-47.7	33.6	38.5	-22.2	51.8	92.2	-13.8	65.8	-87.7
Sciences	71.9	103.5	-80.4	26.1	-14.2	-71.7	-14.1	-111.7	-15.0	10.5	-59.3	4.5
Human	25.2	53.6	-36.4	41.5	-25.5	-9.5	25.9	-87.0	-42.3	-63.2	-21.7	74.9
Medic	-39.6	-69.4	34.7	-20.6	8.2	40.5	7.3	122.8	-18.0	47.9	21.1	-0.7
Odont	-3.1	-5.6	2.7	-1.9	0.9	3.0	0.3	10.2	-1.1	4.6	1.6	-0.8
Pharmac	-14.1	-21.7	13.1	-3.9	1.2	15.6	5.1	31.7	-5.8	4.5	9.0	5.4
Techn	14.0	19.8	-15.3	4.2	-2.2	-14.4	-3.7	-21.4	-1.4	3.1	-11.3	-1.3
	medo	**medp**	**tech**	**mman**	**teach**	**intp**	**stea**	**ocle**	**ebus**	**fore**	**skil**	**sski**
Law	1.1	-23.6	-68.3	-8.1	-143.2	49.3	121.9	-21.1	-5.5	-52.8	-44.3	-62.8
Sciences	-104.3	-19.1	28.2	-7.4	44.7	-46.3	-60.2	6.0	-38.2	35.2	42.1	60.5
Human	-58.9	6.0	43.6	29.9	97.8	-8.8	-10.6	66.4	50.7	55.6	78.0	52.0
Medic	129.6	27.2	-9.8	-10.8	-13.7	11.8	-24.4	-40.4	-2.6	-35.2	-63.4	-46.7
Odont	10.5	2.0	-1.1	-1.2	-1.9	0.8	-2.2	-3.9	-0.9	-3.2	-5.7	-4.0
Pharmac	34.7	8.5	-1.6	0.7	0.2	6.7	-0.7	-4.3	7.4	-7.0	-12.1	-12.4
Techn	-20.8	-4.4	4.4	-2.1	6.1	-9.0	-11.1	-0.1	-8.9	5.8	7.0	11.1
	mine	**fish**	**unsk**	**dome**	**arts**	**Clrg**	**arpo**	**othe**	**unem**	**nona**		
Law	-112.0	-82.1	-75.6	-17.1	103.3	-142.9	24.9	-31.8	40.1	75.5		
Sciences	72.7	52.6	74.4	10.5	-75.6	-24.9	-2.0	37.8	-29.0	-19.1		
Human	76.3	76.5	69.6	85.3	38.9	50.5	30.5	21.9	25.5	36.0		
Medic	-41.8	-45.4	-62.8	-61.0	-36.1	79.0	-38.0	-26.3	-22.1	-62.0		
Odont	-3.8	-4.2	-5.4	-5.8	-3.7	5.8	-3.4	-2.1	-2.2	-5.5		
Pharmac	-10.6	-9.4	-16.0	-7.8	2.9	24.1	-5.8	-7.9	0.1	-8.9		
Techn	12.7	8.8	13.6	0.5	-15.4	-7.7	-0.7	7.2	-6.1	-3.8		

for humanities and in a smaller way for pharmacy. As a matter of fact we see that the inner products among **Human** and all the occupational categories are the highest negative, that means the associations with females from every occupational category are the highest positive. The opposition on the second axis shows that males, whose fathers belong to low-middle occupational category **lfar, farm, mine, teach, fish, etc..**, prefer maths and physics **Sciences**, while males from high-middle class **indu, lsoc, stea, arts, etc..** prefer law study **Law** and in a smaller way medicine **Medic**. The biplot from the SVD of D_1 allows to point out the positive associations for males between the **indu** category of the explicative variable and the **Law** (=102.8) category of the responce variable, between the category **lfar** and the category **Sciences** (=74.7). While the association for males between **indu** and **Human** is negative (=-171.7).

The biplot from the singular value decomposition of D_2 portray the general behaviour of the french students apart from sex. Taking into account the positive sign of the coefficients c_{k2}, we see that the **mine** category is negatively associated with **Law** (=-112.0) and **Medicine** (=-41.8) and positevely with **Sciences** (=72.7) and **Human** (=76.3). The category **ppro** is positively associated with **Law** (=51.7) and **Medic** (=122.8) categories and negatively with **Sciences** (=-111.7) and **Human** (=-87.0).

In general we see that students who belong to low-middle class **farm, mine, lfar, fish, teach, etc..** attend humanities and sciences studies **Human, Sciences** and students from high-middle class **indu, lsoc, ppro, uman, etc..** prefer law and medicine studies **Law, Medic**. At the same time we observe that the second axis divide the society in another direction. Students whose father are trader, manufacturer etc.., attend law study and students whose father belong to medical or scientific occupation attend prevalentely medicine.

9 Conclusion

We have shown that Nonsymmetric Correspondence Analysis for three-way data table contributes to studying the dependence analysis in three-way contingency table as well as to analysing the partial (the first and second order) interactions among the variables (Lancaster's additive decomposition, 1951). The next work could be oriented to develop an integrated approach of NSCA3 with descriptive methodology (logit-linear models). Other prospective of research are related with the possibility to consider different tridimensional association measures (D'Ambra & Lombardo 1993) as well as to check the stability of three-way analysis (Kroonenberg & Snyder 1991). ————————————————————————-

This paper was supported by Central National Reaserch (CNR) for the research project on "....", responsible Prof. L. D'Ambra.

10 Implementation

Three-way Nonsymmetric Correspodence Analysis has been programmed by the first author in S-Plus (Becker, Chambers & Wilks, 1988). The implementation of the program was easy thanks to the availability of most algorithms which Prof. A. Carlier developed for three-way Correspondence Analysis. The technical basis for the algorithms can be found in Kroonenberg (1983, Tucker3 model).

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