30 November 1983

STATEMENT OF SOME CURRENT RESULTS ABOUT THREE-WAY ARRAYS.

Joseph B Kruskal

max rank(I x J x K) means the maximum possible rank of an I x J x K array. If $I \le J \le K$, max rank satisfies

$$IJK/(I+J+K-2) \le \text{max rank}(I \times J \times K) \le IJ,$$

and some exact values are:

```
2 x 2 x K: for K=(1,2,3,>=4),          max rank=(2,3,3,4);
2 x 3 x K: for K=(1,2,3,4,5,>=6),      max rank=(2,3,4,5,5,6);
2 x 4 x K: for K=(1,2,3,4,    7,>=8),  max rank=(2,4,5,6,    7,8);
3 x 3 x K: for K=(1,2,3,     >=9),     max rank=(3,4,5,     9).
```

The geometrical arrange-ent of arrays of different rank within the 8-dimensional space of 2 x 2 x 2 arrays has been worked out in considerable detail.

Most difficult cases are circled.

October 26, 1983
Wednesday, 1:20pm MH Room 6H-404

RANK AND GEOMETRY OF THREE-DIMENSIONAL MATRICES

Joseph B Kruskal (Dept 11214)

The concept of rank extends naturally to 3-dimensional matrices, and the extension has applications to complexity theory and statistics. Calculating rank is difficult even for tiny matrices. We present new results up to 3 x 3 x 3. Max 2 x 2 x 2 rank is 3; surprisingly, both rank 2 and rank 3 occur with positive measure. Their geometrical arrangement is described.

RANK & GEOMETRY of THREE-DIMENSIONAL MATRICES

Joseph B Kruskal

26 Oct 1983

Rank of a matrix is a major mathematical concept. There is a natural generalization to n-dim matrices or tensors having rank [in different sense] n.

It is so natural that it has arisen at least 4 times, apparently independently:

1. computational complexity - Strassen (1968) and many subsequent papers
2. psychometrics/statistics - Carroll & Chang ('70), Harshman (1970), etc.
3. abstract algebra - Watkins (1976)
4. differential geometry? - [reference mislaid]

I will present it as a concept of pure mathematics.

The most elegant approach defines rank for <u>sets</u> of n-dim. matrices.

All results for <u>real</u> <u>numbers</u> $\mathbb{R}$ only.

First I will define dim and rank in a formal way. Then I will connect them to <u>matrix rank, etc</u>

"dim" has 2 meanings
- as above
- as in "N-~~dim~~ ~~matrix~~"

"rank" has 2 meanings
- as above
- as in "rank ~~n~~ tensor"

Use N-way array or N-way tensor instead

N-way array has elements $X_{i_1 \cdots i_N}$

Vector is 1-way array. Matrix is 2-way array.

$\mathbb{R}^{I}$ = vectors (I-tuples) = I arrays

$\mathbb{R}^{I \times J}$ = $I \times J$ matrices = $I \times J$ arrays

$\mathbb{R}^{I \times J \times K}$ = $I \times J \times K$ arrays

Size of array is I or (I,J) or (I,J,K) etc

Arrays (of fixed size) form a <u>vector space</u>.

$\mathcal{V}$ is a set of arrays (of same size)

Def Linear span of $\mathcal{V}$ is set of all linear combinations of $\mathcal{V}$
= linear space (i.e., vector space) generated by $\mathcal{V}$
= smallest linear space containing $\mathcal{V}$.

If $\mathcal{V}$ is a linear space, then
$\mathcal{V}$ = linear span of $\mathcal{V}$

Def. $\dim(\mathcal{V})$ = min # of arrays whose linear span contains $\mathcal{V}$.

If $\mathcal{V}$ is a linear space, then $\dim(\mathcal{V})$ agrees with usual meaning.

If $\mathcal{V}$ consists of n indep elts, $\dim(\mathcal{V}) = n$

If matrix $m \neq 0$, array $X \neq 0$,
$$\dim(m) = 1, \quad \dim(X) = 1.$$

$\dim(\mathcal{V}) = \dim(\text{linear span of } \mathcal{V})$

Def An N-way array is an outer product array if it is the outer product of N vectors.

Meaning (Meaning of —)

Outer product matrix: $m_{\_\_} = a_{\_} \otimes b_{\_}$,

$$m = \begin{bmatrix} a_1 \\ \vdots \\ \vdots \end{bmatrix} [b_1 \cdots], \qquad m_{ij} = a_i b_j$$

Dyad, dyadic matrix, "multiplication table", rank-1 matrix, ...

Outer product 3-way array $X_{\_\_\_} = a_{\_} \otimes b_{\_} \otimes c_{\_}$

$$X_{ijk} = a_i b_j c_k$$

Triad, triadic array, rank-1 array

Outer product vector: ANY vector!
Vacuous restriction.

$\mathcal{V}$ set of arrays (of same size)

<u>Def</u> rank($\mathcal{V}$) = min # of outer product arrays whose linear span contains $\mathcal{V}$

Compare to def of "dim"

<u>Fact</u>: rank($\mathcal{V}$) $\geq$ dim ($\mathcal{V}$)

<u>Fact</u>: rank($\mathcal{V}$) = rank (linear span of $\mathcal{V}$)

If $\mathcal{V}$ consists of <u>single</u> <u>matrix</u>, this def agrees with classical meaning of rank.

If $\mathcal{V}$ consists of <u>vectors</u> (1-way arrays),

$$\text{rank}(\mathcal{V}) = \dim(\mathcal{V}),$$

because all vectors are "outer product vectors".

If $\mathcal{V}$ is empty or $\mathcal{V}$ consists of 0 array, rank ($\mathcal{V}$) = 0, and conversely.

Suppose X is $I \times J \times 1$ array, & $X_{ij1} = m_{ij}$.
Then rank (X) = rank (m); and similar result holds in general.

SOME EXAMPLES

$\mathcal{V}$	dim	$\leq$	rank	generating set of outer product mat.s
$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	1		1	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$
$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	1		2	$\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$
$\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}$	2		2	$\begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}1&-1\\-1&1\end{bmatrix}$
$\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\-1&0\end{bmatrix}$	2		3	$\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\-1&-1\end{bmatrix}$
Any 2 2×2 matrices	≤ 2		≤ 3	
Any 3 ...	≤ 3		≤ 3 same!	
Any 4 or more ...	≤ 4		≤ 4	$\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$

<u>Classical</u>: Matrix also has row-rank, col-rank

$\text{row-rank}(m) \stackrel{\text{def}}{=} \dim(\text{rows of } m)$

$\text{col-rank}(m) \stackrel{\text{def}}{=} \dim(\text{cols of } m)$

<u>Lemma</u>: row-rank = col-rank = rank
(over any field; they may differ over a ring). $\square$

We incorporate this as follows.

<u>Def</u>: If X is N-way array of size $(I_1, \cdots, I_N)$ and elts $X_{i_1 \cdots i_N}$, then the I_n n-subarrays of X (for $n=1$ to N) are

$X_{-\cdots- i_n -\cdots-}$ with $i_1 = 1$ to I_n.

1-subarrays of matrix m are rows m_{i-}
2- " " " " " cols m_{-j}

1-subarrays of 3-way array X are 1-slabs X_{i--}
2- " " " " " 2-slabs X_{-j-}
3- " " " " " 3-slabs X_{--k}

<u>Def</u> n-rank(X) = rank(the n-subarrays of X).

Note: We have <u>not</u> defined for <u>sets</u> of arr . .

For matrix m,

1-rank(m) = rank(1-subarrays) = rank(rows)

= dim(rows) = row-rank

2-rank(m) = col-rank(m).

<u>Lemma</u>: For any N-way X,

1-rank = 2-rank = ... = N-rank = rank.

<u>Could</u> define n-rank for sets of arrays,
but this result would not hold.

<u>Def</u>: n-dim(X) = dim(the n-subarrays of X)

For matrix m,

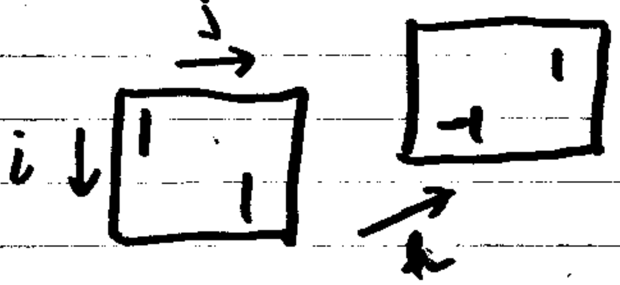
$$\begin{array}{ccccc} \text{rank} & = & \text{1-rank} & = & \text{2-rank} \\ & & \| & & \| \\ & & \text{1-dim} & & \text{2-dim} \end{array}$$

For 3-way array X,

$$\begin{array}{ccccccc} \text{rank} & = & \text{1-rank} & = & \text{2-rank} & = & \text{3-rank} \\ & & \vee\| & & \vee\| & & \vee\| \\ & & \text{1-dim} & & \text{2-dim} & & \text{3-dim.} \end{array}$$

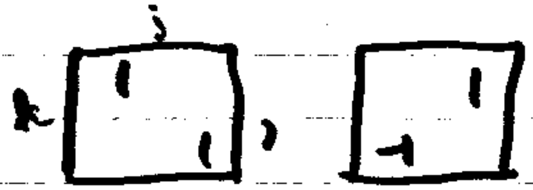
EXAMPLE - Array of size (2, 2, 2)

Omitted elements = 0



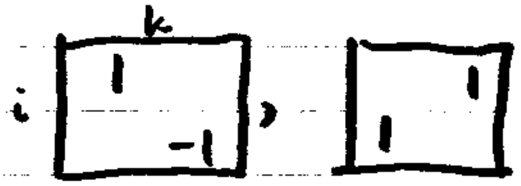
3-slabs

3-dim = 2
3-rank = 3



1-slabs

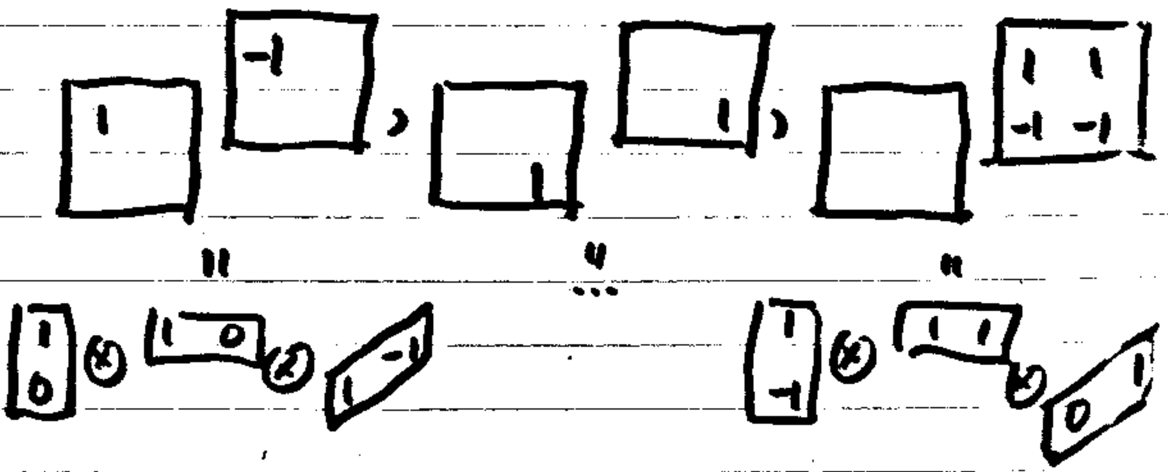
1-dim = 2
1-rank = 3



2-slabs

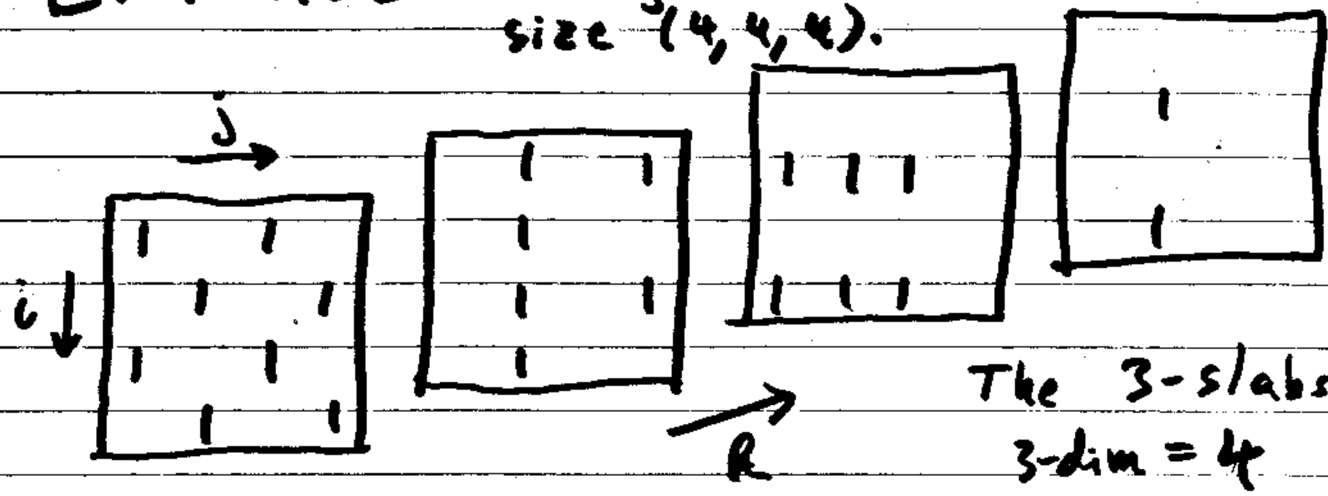
2-dim = 2
2-rank = 3

rank = 3

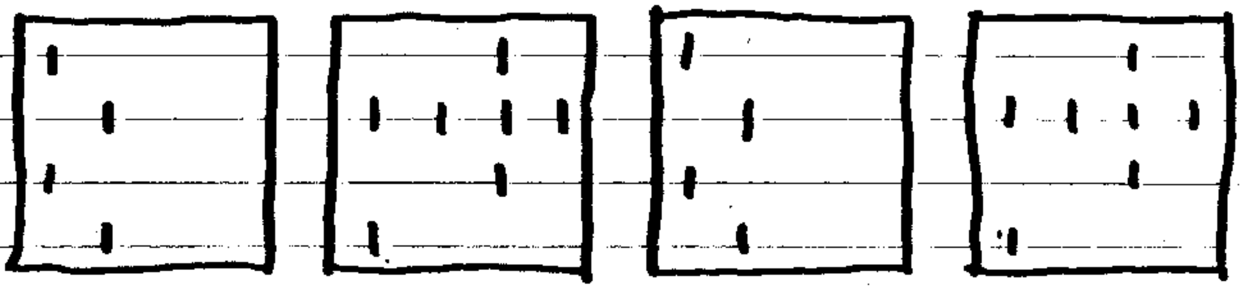


rank	=	1-rank	=	2-rank	=	3-rank
3		3		3		3
		$\geq$		$\geq$		$\geq$
		1-dim		2-dim		3-dim
		2		2		2

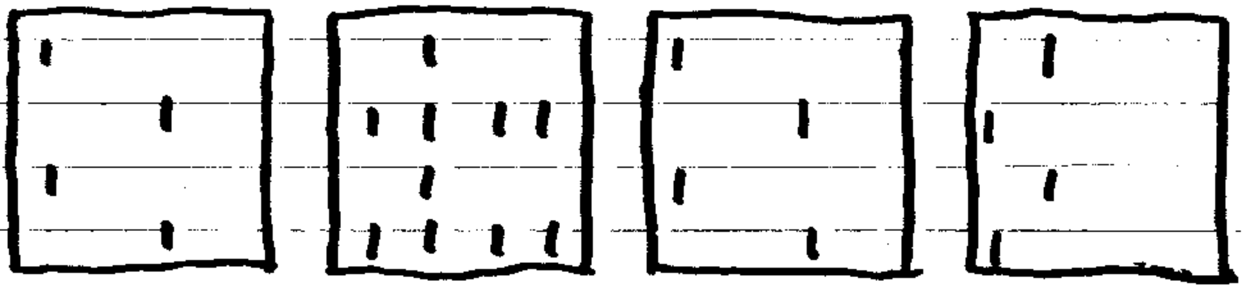
EXAMPLE - Array of size (4, 4, 4).



The 3-slabs
3-dim = 4



The 1-slabs
1-dim = 2



The 2-slabs
2-dim = 3

rank =	1-rank =	2-rank =	3-rank
4	4	4	4
	VII	VII	VII
	1-dim	2-dim	3-dim
	2	3	4

EXAMPLE – from Strassen algorithm

$$\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$$

Then $w_1 = u_1 v_1 + u_2 v_3$

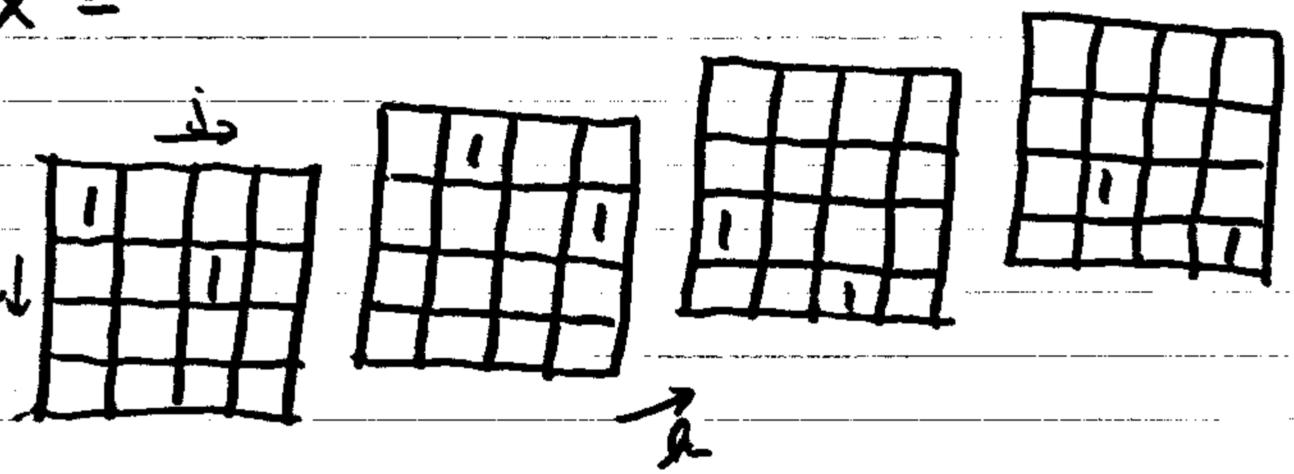
$w_2 = \cdots$, etc

Define X_{ijk} to be the coeffs in

$$w_k = \sum_i \sum_j X_{ijk}\, u_i v_j$$

Then

$X =$



1-dim = 2-dim = 3-dim = 4

rank = 7

rank(m) = min # of outer product matrices whose linear span contains m

= min R for which it is possible to wr...

$$m_{ij} = \sum_{r=1}^{R} a_{ir} b_{jr}$$ { Each r is one outer product

or $m = {}_I a_R b_J^T$

= min R for which ${}_I m_J = {}_I a_R b_J^T$ possible

rank(X) = min R for which it is possible to write

$$X_{ijk} = \sum_{r=1}^{R} a_{ir} b_{jr} c_{kr}$$ "Triple product"

or $\underset{(I,J,K)}{X} = [{}_I a_R, {}_J b_R, {}_K c_R]$

<u>Decomposition</u> means $m = ab^T$ or

$X = [a, b, c]$

Rank & uniqueness - Kruskal (1977)

Comment - Two applications areas concerned with 3-way arrays.

Sometimes rank(X) is central.

Sometimes decomposition is central.

Statistical PARAFAC model is based on least-squares-fitting by arrays of bounded rank.

Current work grew out of questions this stimulated. -- gave talk here in June.

Though this concept of rank goes back a few years, & can be found several places in the literature, this concise unified definition is new.

SOME PREVIOUS RESULTS ON RANK

For $I \times J \times K$ array:

$$\left.\begin{matrix} \text{1-dim} \\ \text{2-dim} \\ \text{3-dim} \end{matrix}\right\} \leq \text{rank} \leq \begin{cases} (\text{1-dim})(\text{2-dim}) \leq IJ \\ (\text{1-dim})(\text{3-dim}) \leq IK \\ (\text{2-dim})(\text{3-dim}) \leq JK \end{cases}$$

$$\text{rank} \leq \begin{cases} \#\text{ of non-zero 1-lines } X_{-jk} \\ \# \quad " \quad " \quad " \text{ 2-lines } X_{i-k} \\ \# \quad " \quad " \quad " \text{ 3-lines } X_{ij-} \end{cases}$$

Will need in proof later

[Frobenius (1911): rank(um) + rank(mw) − rank(umw) $\leq$ rank(m)]

<u>Define:</u> matrix $\circledast_1$ ARRAY: $(u \circledast_1 X)_{ijk} = \sum_{\bar{\imath}} u_{i\bar{\imath}} X_{\bar{\imath}jk}$

$(u,v,w) \circledast X = u \circledast_1 (v \circledast_2 (w \circledast_3 X))$

$$\text{1-dim}(u \circledast_1 X) + \text{rank}(v \circledast_2 w \circledast_3 X) - \text{1-dim}((u,v,w) \circledast X) \leq \text{rank}(X)$$

If 1-slabs of X are indep., then

$$\min_{\substack{\text{all } u \text{ with} \\ \text{lin. indep. rows}}} \text{rank}(u \circledast_1 X) + \text{1-dim}(X) - \text{1-dim}(u) \leq \text{rank}(X)$$

---

etc., etc., etc.

Lots of results — but lots of holes

Strassen array: rank 7; general results 6-8

HOLES ARE LARGE

See soon we can prove <u>existence</u> of high rank arrays but have very <u>little</u> ability to prove that <u>individual</u> arrays have that rank.

SOME MOTIVATING QUESTIONS:

What is max possible rank for $I \times J \times K$ array?

How are arrays of each rank arranged geometrically in IJK-space?

Methods for calculating rank, or at least bounds on rank (for given array)?

Counting Parameters (rigorous argument)

$I \times J \times K$ array has IJK d.f.

Triple-product provides mapping

$$(A, B, C) \to [A, B, C]$$

from $\mathbb{R}^{I \times R} \oplus \mathbb{R}^{J \times R} \oplus \mathbb{R}^{K \times R} \to \mathbb{R}^{I \times J \times K}$.

Since this mapping is smooth, it cannot be space-filling unless

$$(IR + JR + KR) \equiv (I+J+K)R \geq IJK.$$

Thus

$$\text{max rank} \geq \text{min space-filling rank} \geq \frac{IJK}{I+J+K}.$$

Improvement: If Λ, M, N are diagonal and $\Lambda M N = I$, then

$$[A\Lambda, BM, CN] = [A, B, C].$$

(Λ, M, N) has $3R$ parameters; but only $2R$ d.f. Then the mapping cannot be space-filling unless

$$(I+J+K)R - 2R \equiv (I+J+K-2)R \geq IJK.$$

Thus
$$\text{min space-filling rank} \geq \frac{IJK}{I+J+K-2}$$

For $I \times I \times I$ arrays

max rank $\geq$ min space-filling rank $\geq \dfrac{I^3}{3I-2}$

I	$I^3/(3I-2)$
1	1
2	2
3	3.9 $\to$ 4
4	6.4 $\to$ 7 (compare Strassen array)
5	9.6 $\to$ 10
I	$\approx \frac{1}{3} I^2$

For $I \times I \times K$ max rank $\geq \dfrac{I^2 K}{2I+K-2}$

$I=4$

K	$I^2K/(2I+K-2)$	
1	2.3 $\to$ 3	[4]
2	4	?
3	5.3 $\to$ 6	
$\boxed{4}$	$\boxed{6.4 \to 7}$	
5	7.3 $\to$ 8	
6	8	
7	8.6 $\to$ 9	

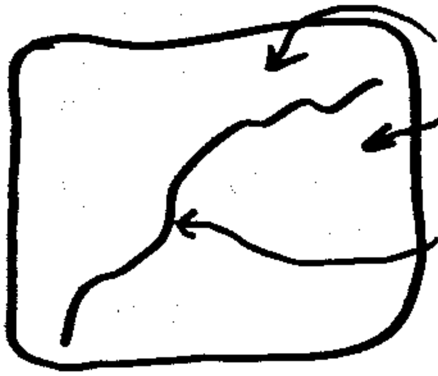
$I=5$

K	
1	[5]
2	
3	
4	
$\boxed{5}$	$\boxed{9.6 \to 10}$
6	
7	

RANK OF $N \times N$ MATRICES (ILLUSTRA)

$\mathcal{M}_N$

N^2-dim



Rank N

Rank $\leq N-1$

determinant $= 0$

Rank N "space-filling" N^2-dim

Lebesgue measure (= generalized volume)

positive

Rank $\leq N-1$ NOT space-filling, (N^2-1)-dim curved manifold

measure 0

Rank is N 'except for a set of measure 0'

Rank is N almost everywhere

ILLUSTRATION
RANK OF N×N MATRICES

$\mathcal{M}$ = set of all N×N matrices

	Rank	Dimension	Lebesgue measure
$\mathcal{M}_N = \mathcal{M}$	$\leq N$	N^2	positive
$\cup\!\!\mid$			
$\mathcal{M}_{N-1}$ (det=0)	$\leq N-1$	N^2-1	0
$\cup\!\!\mid$			
$\mathcal{M}_{N-2}$	$\leq N-2$	N^2-4	0
$\cup\!\!\mid$			
$\mathcal{M}_{N-3}$	$\leq N-3$	N^2-9	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{M}_1$ (dyads)	≤ 1	$2N-1$	0
$\cup\!\!\mid$			
$\mathcal{M}_0$ (zero matrix)	0	0	0

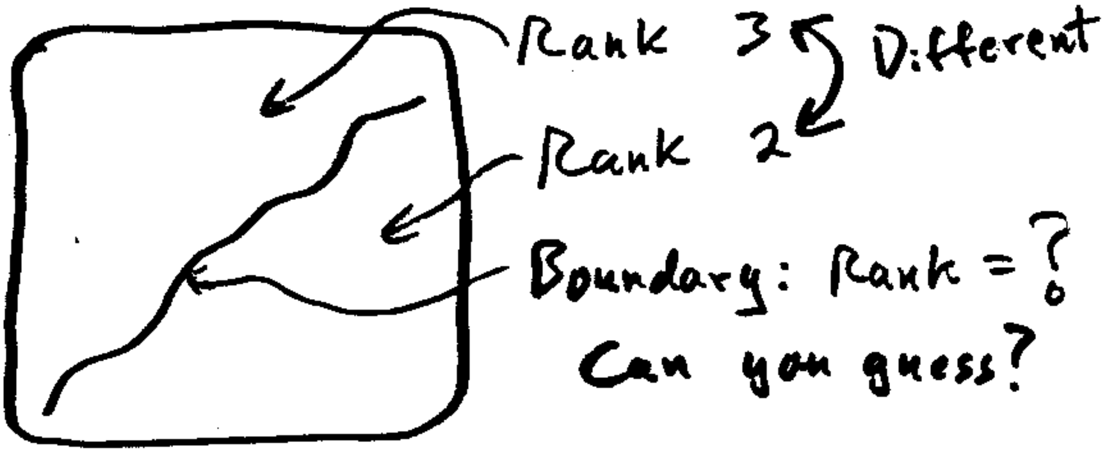
[area, volume, ⋮]

Each $\mathcal{M}_n$ is curved manifold

RANK OF 2×2×2 ARRAYS

a

8-dim



Rank 3 space-filling 8-dim

Rank 2 space-filling 8-dim

Boundary NOT space-filling, 7-dim

Fourth-degree homogeneous polynomial (12 terms)

(akin to determinant of matrix)

positive	rank 3
0	boundary
negative	rank 2

RANK OF 2×2×2 ARRAYS

$\mathcal{A}$ = set of all arrays

	Rank	Dim	Leb. meas
$\mathcal{A}$		8	pos
$\mathcal{A}_2 \cup \mathcal{A}_0 \cup \mathcal{A}_3$	2 ≤3 3	8 7 8	pos 0 pos
$\mathcal{B}_1 \quad \mathcal{B}_2 \quad \mathcal{B}_3$	≤2 ≤2 ≤2	5 5 5	0 0 0
$\mathcal{C}$	≤1	4	0
$\mathcal{D}$ (zero array)	0	0	0

$$\mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_0 \cup \mathcal{A}_3$$

$$\mathcal{A}_0 \supseteq \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$$

$$\mathcal{B}_1 \supseteq \mathcal{C}, \quad \mathcal{B}_2 \supseteq \mathcal{C}, \quad \mathcal{B}_3 \supseteq \mathcal{C}$$

$$\mathcal{C} \supseteq \mathcal{D}$$

$j \rightarrow$, $i \downarrow$, $k \nearrow$

111	121
211	221

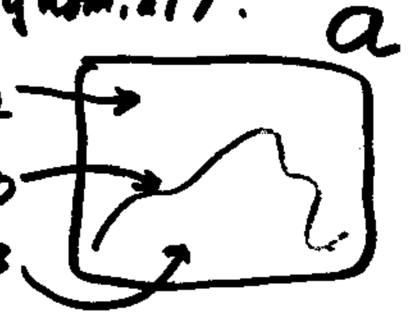
112	122
212	222

gener. "determinant" (4th deg polynomial):

neg : $\mathcal{A}_2$

0 : $\mathcal{A}_0$

pos : $\mathcal{A}_3$



$\mathcal{B}_1 : A_{1--} \sim A_{2--}$

$\mathcal{C} = \mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}_1 \cap \mathcal{B}_3 = \mathcal{B}_2 \cap \mathcal{B}_3$

= outer product arrays

ARRAY SIZE	RANK DISTRIBUTION
$1 \times J \times (K \geq J)$	$0 \quad 1 \quad 2 \quad \cdots \quad J-1 \quad \boxed{J}$
$2 \times 2 \times 2$	$0 \quad 1 \quad \boxed{2}\boxed{3}$
$2 \times 2 \times 3$	$0 \quad 1 \quad 2 \quad \boxed{3}$
$2 \times 2 \times (K \geq 4)$	$0 \quad 1 \quad 2 \quad 3 \quad \boxed{4}$
$2 \times 3 \times 3$	$0 \quad 1 \quad 2 \quad \boxed{3} \quad \boxed{4}$?
$2 \times 3 \times 4$	$0 \quad 1 \quad 2 \quad 3 \quad \boxed{4}$
$2 \times 3 \times (K \geq 6)$	$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \boxed{6}$
$3 \times 3 \times 3$	$0 \quad 1 \quad 2 \quad 3 \quad \boxed{4} \quad \boxed{5} \quad (\cancel{6}!)$ Probably not possible

A FEW PROOFS

Thm: Rank $(2 \times 2 \times 2) \leq 3$.

Proof: Since rank $=$ 1-rank, prove that any set of 2 2×2 matrices has rank ≤ 3.
Let $\mathcal{V}$ be linear span of the set.
Know $\dim(\mathcal{V}) \leq 2$. Prove: $\operatorname{rank}(\mathcal{V}) \leq 3$

Case I: $\dim(\mathcal{V}) \leq 1$: Then $\operatorname{rank}(\mathcal{V}) =$ rank of single 2×2 matrix ≤ 2.

Case II: $\dim(\mathcal{V}) = 2$, every matrix in $\mathcal{V}$ has rank ≤ 1: Then any generating set consists of 2 outer product matrices, so $\operatorname{rank}(\mathcal{V}) \leq 2$.

Case III: $\dim(\mathcal{V}) = 3$, $\mathcal{V} \ni$ matrix of rank 2:
Let $\{a, b\}$ be basis with $\operatorname{rank}(a) = 2$.
Then $a^{-1}\{a, b\} = \{I, a^{-1}b\} = \{I, c\}$

$$= \left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right\} \text{ is a basis.}$$

Let d be a dyad (outer product matrix) which agrees with c in positions $(1,2)$ & $(2,1)$:

$$d^1 = d = \begin{bmatrix} ? & c_{12} \\ c_{21} & ? \end{bmatrix}.$$

E.g., $d_{11} \neq 0$, $d_{22} = \dfrac{c_{12}\, c_{21}}{d_{11}}$.

Then $\left\{ d^1,\ d^2 = \begin{bmatrix} 1 & \\ & \end{bmatrix},\ d^3 = \begin{bmatrix} & \\ & 1 \end{bmatrix} \right\}$

is a dyadic (outer product) generating set for V, because

$$I \equiv \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & \end{bmatrix} + \begin{bmatrix} & \\ & 1 \end{bmatrix}$$

$$c = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} + (d_{11} - c_{11}) \begin{bmatrix} 1 & \\ & \end{bmatrix} + (d_{22} - c_{22}) \begin{bmatrix} & \\ & 1 \end{bmatrix}$$

Thm: rank $(2\times2\times3) = 3$ almost everywhere.

rank ≥ 3 a.e.: The 3 2×2 slices are indep a.e., so rank ≥ 3 a.e..

rank $(2\times2\times3) \leq 3$:

Def: 2 dyads $u_1 \otimes v_1$ and $u_2 \otimes v_2$ are compatible if **either** $u_1 \sim u_2$ **or** $u_2 \sim v_2$.

Lemma: 3 pairwise incompatible 2×2 dyads are indep.

Proof Let the dyads be $u_i \otimes v_i = \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} [v_{i1} \ v_{i2}]$

Consider a possible dependence,

$$0 = \sum_1^3 \alpha_i \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} [v_{i1} \ v_{i2}] = \sum \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} \alpha_i [v_{i1} \ v_{i2}]$$

$$= \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{bmatrix}$$

$$\equiv \bar{u}\, \bar{\alpha}\, \bar{v}$$

Case 0: No $\alpha_i \neq 0$. The eqn. is not a linear dependence.

Case 1: Just one $\alpha_i \neq 0$, say α_1. Then $u_1 \otimes v_1 = 0$, so either $u_1 = 0$ or $v_1 = 0$, say u_1. Then $u_1 \sim u_2$, contrary to hypothesis.

Case 2: Just two $\alpha_i \neq 0$, say α_1 & α_2. Then eqn reduces to

$$0 = \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \bar{\bar{u}} \, \bar{\bar{\alpha}} \, \bar{\bar{v}}$$

Now $\operatorname{rank}(\bar{\bar{\alpha}}) = 2$, and also

$$\left.\begin{matrix} \operatorname{rank}(\bar{\bar{u}}) = 2 \\ \operatorname{rank}(\bar{\bar{v}}) = 2 \end{matrix}\right\} \text{ by incompatability hypothesis.}$$

Thus $\operatorname{rank}(\bar{\bar{u}} \, \bar{\bar{\alpha}} \, \bar{\bar{v}}) = 2$, so $\bar{\bar{u}} \, \bar{\bar{\alpha}} \, \bar{\bar{v}} \neq 0$

Case 3: All $\alpha_i \neq 0$. Then $\operatorname{rank}(\bar{\alpha}) = 3$, $\operatorname{rank}(\bar{u}) = \operatorname{rank}(\bar{v}) = 2$ by incompatability. Frobenius Theorem yields

$$\operatorname{rank}(\bar{u}\,\bar{\alpha}\,\bar{v}) \geq \operatorname{rank}(\bar{u}\,\bar{\alpha}) + \operatorname{rank}(\bar{\alpha}\,\bar{v}) - \operatorname{rank}(\bar{\alpha})$$

$$= 2 + 2 - 3 = 1,$$

so $\bar{u}\,\bar{\alpha}\,\bar{v} \neq 0$. This proves lemma.

<u>Proof that rank $(2\times 2\times 3) \leq 3$.</u>

rank $=$ 3-rank, so it is enough to show that any $\mathcal{V}$ generated by 3 2×2 matrices has rank ≤ 3. Since $R^{2\times 2}$ is 4-dim, $\exists\, m \in R^{2\times 2}$ orthog to $\mathcal{V}$.

If $a, b \in R^{2\times 2}$, Their dot product

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} b_{ij} = tr(a b'),$$

where $tr(c) = \text{trace}(c) = \sum c_{ii}$.

Thus $\mathcal{V} = \{ a \mid tr(a m') = 0 \}$.

It will suffice to find 3 indep dyads in $\mathcal{V}$

<u>Case 1</u>: m is <u>not</u> a dyad.

For any non-zero column vector u, $u'm' \neq 0$,

so $\exists$ col vector $v = v(u)$ (unique up to scale) $\ni$ $u'm'v = 0$;

and $vu' \in \mathcal{V}$ because

$$tr(vu'm') = tr(u'm'v) = 0.$$

Pick any 3 u_i which are pairwise indep.
Then $u_i' m$ are pairwise indep, so v_i are also. By the lemma, $v_i u_i'$ are indep.

<u>Case 2:</u> m <u>is</u> a dyad, $m = x \otimes y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} [y_1, y_2]$

$m' = (xy')' = yx'$.

$\mathcal{V} = \{ a \mid tr(am') = 0 \} = \{ a \mid tr(ayx') = 0 \}$.

Set of dyads in $\mathcal{V}$

$= \{ uv' \mid 0 = tr(uv'yx') = tr(x'u\, v'y)$
$= x'u\, v'y = (x \cdot u)(v \cdot y) \}$

Thus $uv' \in \mathcal{V}$ iff u orthog to x <u>or</u>
v orthog to y

Let $x^\perp$ & $y^\perp$ be orthog to x & y, respectively.

Then $x^\perp \otimes y^\perp$, $x \otimes y^\perp$, $x^\perp \otimes y$
are dyads in $\mathcal{V}$. Indep. follows by simple direct argument.

TOOLS I NEED THAT MAY BE KNOWN

geometry of matrices

e.g., in $\mathbb{R}^{I \times J}$, what can we say about intersection of p-dim subspaces & $\mathcal{M}_q$ (matrices of rank $\leq q$)

algebraic geometry