Typical Tensorial Rank*

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ABSTRACT

Upper bounds on the typical rank $\underline{R}(n,m,l)$ of tensors (= maximal border rank = rank of almost all tensors) of a given shape (n,m,l) are presented. These improve previous results by Atkinson and Lloyd. For cubic shape tensors the typical rank is determined exactly: $\underline{R}(n,n,n)=\lceil n^3/(3n-2)\rceil$ $(n\neq 3)$.

1. INTRODUCTION

The problem of finding optimal computations for bilinear forms, in the framework of algebraic complexity theory, leads to the notion of tensorial rank: Given finite dimensional vector spaces U, V, and W over some field k and $t\in U\otimes V\otimes W$. The task consists in finding a decomposition of t into triads,

$$(1.1)\qquad t=\sum_{\rho=1}^{r} u_\rho\otimes v_\rho\otimes w_\rho \qquad (u_\rho\in U,\quad v_\rho\in V,\quad w_\rho\in W),$$

with minimal possible r. The least r for which such a representation exists is called the rank of t, $\operatorname{rk} t$. (See [5,8,14,20] for motivation and background.)

The problem simplifies if the underlying field k is algebraically closed. In the sequel this is always assumed. In spite of the simplicity of the formulation, a complete solution of the problem is only known for 2-slice tensors (e.g. $\dim W=2$) in terms of the Kronecker-Weierstrass normal form ([9,13] and

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also [7] in special cases). Unfortunately this case (as well as the trivial case $\dim W = 1$) is not typical for the general case. So at present the problem is far away from being solved.

An interesting subproblem in this connection is the question for the rank of a general tensor, or the question for the maximum value of the rank of tensors in a tensor product space $U \otimes V \otimes W$ of shape (n, m, l). Let

$$R(n, m, l) = \max\{\operatorname{rk} t : t \in U \otimes V \otimes W\},$$

where $n = \dim U$, $m = \dim V$, and $l = \dim W$. Several authors have noted that almost all tensors in $U \otimes V \otimes W$ have the same rank $\underline{R}(n, m, l)$, possibly smaller than $R(n, m, l)$ (depending on the shape). $\underline{R}$ is called the typical rank and can be described by approximation: Let

$$X_r = \{t : \underline{\operatorname{rk}}\, t \leqslant r\}, \tag{1.2}$$

where $\underline{\operatorname{rk}}$ denotes the border rank in the sense of Bini et al. [4] and Schönhage [18]. Then

$$\underline{R}(n, m, l) = \min\{r : X_r = U \otimes V \otimes W\} = \max\{\underline{\operatorname{rk}}\, t : t \in U \otimes V \otimes W\}.$$

One has the lower bounds

$$R(n, m, l) \geqslant \underline{R}(n, m, l) \geqslant \frac{nml}{n + m + l - 2} \tag{1.3}$$

[6, 9, 12], and Atkinson and Stephens [3] show the upper bound

$$R(n, m, l) \leqslant \lfloor l/2 \rfloor n + m \qquad (m \leqslant n).$$

An equivalent concept to describe X_r or $\underline{R}$ is as follows: Consider $U \otimes V \otimes W$ as an affine space in the sense of algebraic geometry. Then

$$X_r = \text{Zariski closure of } \{t : \operatorname{rk} t \leqslant r\} \tag{1.4}$$

(cf. [1]), and $\underline{R}$ is the common rank of all tensors in some nonempty Zariski open subset of $U \otimes V \otimes W$. Therefore $\underline{R}$ may be regarded as the more natural quantity than R. If $R \neq \underline{R}$, then $\operatorname{rk} t = R$ only holds in some lower dimensional subvariety of $U \otimes V \otimes W$ (which is of Lebesgue measure zero, in case of $K = \mathbf{C}$).

A first step in estimating $\underline{R}$ from the above has been taken by Atkinson and Lloyd [2]. They formulate, for $n = m$, open conditions (i.e., conditions fullfilled in some nonempty Zariski open part of $U \otimes V \otimes W$) under which a decomposition (1.1) of length $r = \lceil l/2 \rceil n$ exists, i.e.,

$$\underline{R}(n, n, l) \leqslant \lceil l/2 \rceil n.$$

A completely different way to find or to estimate the typical rank is by considering the representation (1.1) in first order approximation, an approach independently taken by V. Strassen [19] and the author [16]. In the present paper we determine the dimension of the tangent space of X_r in a general point by means of the tangential mapping of the triadic decomposition (1.1) to find the dimension of X_r in certain cases.

For the typical rank we show

$$\underline{R}(n, m, l) = \frac{nml}{n + m + l - 2} + O(l) \qquad (n \leqslant m \leqslant l).$$

From the complexity point of view, the most interesting tensors are of cubic shape (e.g., tensors corresponding to the multiplication in a k-algebra). For cubic shapes we give the precise value of the typical rank:

$$\underline{R}(n, n, n) = \left\lceil \frac{n^3}{3n - 2} \right\rceil \qquad (n \neq 3).$$

Moreover, we obtain the following surprisingly simple formula for the dimension of X_r:

$$\dim X_r = \min\{ r(3n - 2),\ n^3 \} \qquad (n \neq 3).$$

These results are based upon corresponding results for quasicubic shapes $(n, n, n + 2)$ and $(n, n, n - 1)$ by V. Strassen [19].

Throughout this paper we use the following notation:

$\mathbb{Z}$	the integers;
$[a, b]$	$= \{a, a+1, \ldots, b\}$, $a, b \in \mathbb{Z}$;
$\lvert M \rvert$	cardinality of the set M;
$M \sqcup N$	disjoint union of M and N;
$\operatorname{span} M$	linear span of M;
T/M	factor space $(T + M)/M \cong T/(M \cap T)$, T, M linear subspaces of some linear space.

2. THE TANGENTIAL MAPPING OF THE TRIADIC DECOMPOSITION

In this section we develop the techniques to find the dimension of X_r. Let k be an algebraically closed field, and U, V, and W finite-dimensional vector spaces over k of dimensions n, m, and l. The triple (n, m, l) we call the shape of the tensor product space $U \otimes V \otimes W$. The Segre variety ([11, 17])

$$\mathbb{S} = \mathbb{S}(U \otimes V \otimes W) = \{ u \otimes v \otimes w : u \in U,\ v \in V,\ w \in W \}$$

is an $(n + m + l - 2)$-dimensional subvariety of $U \otimes V \otimes W$ and regular (= smooth) except at 0. The triadic decomposition (1.1) induces a morphism of affine varieties

$$\varphi_r : \mathbb{S}^r \to X_r, \qquad (t_1, \ldots, t_r) \mapsto \sum_{\rho \leqslant r} t_\rho .$$

By the definition of φ_r

$$\{ t : \operatorname{rk} t \leqslant r \} = \operatorname{im} \varphi_r ;$$

therefore $\{ t : \operatorname{rk} t \leqslant r \}$ is irreducible and constructible in the sense of algebraic geometry (by a theorem of Chevalley, [17, p. 37]). Moreover, $\{ t : \operatorname{rk} t \leqslant r \}$ and X_r are defined over the prime field of k, since $\mathbb{S}$ is, and φ_r only sums. The closure X_r is easier to describe than the constructible set $\{ t : \operatorname{rk} t \leqslant r \}$ (since it is already given by algebraic equations). X_r is an irreducible and homogeneous variety, and

$$\dim X_r \leqslant r \dim \mathbb{S} = r(n + m + l - 2).$$

Almost all tensors in X_r have rank $\leqslant r$, since the image $\operatorname{im} \varphi_r \subseteqq X_r$ contains an open dense part of its closure (cf. [17]). We have the ascending chain

$$\mathbb{S} = X_1 \subsetneqq X_2 \subsetneqq \cdots \subsetneqq X_{\underline{R}(n, m, l)} = U \otimes V \otimes W .$$

In the present paper we consider the question whether the dimensions of these varieties increase maximally, i.e., $\dim X_r - \dim X_{r-1} = \dim \mathbb{S} = n + m + l - 2$ $[r < \underline{R}(n, m, l)]$.

Following Strassen [19], we call r *small* if $\dim X_r = r(n + m + l - 2)$, and *large* if $X_r = U \otimes V \otimes W$, i.e., $\dim X_r = nml$. $U \otimes V \otimes W$ or its shape (n, m, l) is called *good* if always $\dim X_r = \min\{r(n+m+l-2), nml\}$; it is called *perfect* if in addition $nml/(n+m+l-2)$ is an integer. Obviously the following relations hold:

$$\begin{aligned}
&(2.1) \qquad & r \text{ small} \quad &\Rightarrow \quad r-1 \text{ small};\\
&& r \text{ large} \quad &\Rightarrow \quad r+1 \text{ large};\\
&& (n,m,l) \text{ good} \quad &\Leftrightarrow \quad \left\lceil \frac{nml}{n+m+l-2} \right\rceil \text{ large and } \left\lfloor \frac{nml}{n+m+l-2} \right\rfloor \text{ small};\\
&& (n,m,l) \text{ perfect} \quad &\Leftrightarrow \quad \underline{R}(n,m,l) = \frac{nml}{n+m+l-2}.
\end{aligned}$$

For instance, the 2-slice shape

$$(n,n,2) \text{ is perfect}$$

[9], and "too long" shapes

$$(n,m,l) \text{ are not good} \qquad (l \geqslant nm \geqslant 1).$$

Now let $\mathbf{t} = (t_1,\dots,t_r) \in \mathbb{S}^r$. We consider the tangential mapping of φ_r (cf. [17, p. 41])

$$d_{\mathbf{t}\varphi_r}: T_{\mathbf{t}\mathbb{S}^r} \to T_{\varphi_r(\mathbf{t})X_r},$$

(where T_{aB} is the tangent space to B at the point $a \in B$) to find the dimension of X_r via the image of $d_{\mathbf{t}\varphi_r}$. We have

$$(2.2) \qquad \operatorname{im} d_{\mathbf{t}\varphi_r} = \sum_{\rho \leqslant r} T_{t_\rho \mathbb{S}},$$

since φ_r only sums, and $T_{\mathbf{t}\mathbb{S}^r} = \prod_{\rho \leqslant r} T_{t_\rho \mathbb{S}}$. Thus we first determine the tangent space $T_{t\mathbb{S}}$ for a triad t. Let $D \subseteqq U$, $E \subseteqq V$, and $F \subseteqq W$ be subspaces. (In the sequel we reserve the letter D to denote subspaces of U, E for subspaces of V, and F for subspaces of W.) We put for some triad $t = u \otimes v \otimes w \in U \otimes V \otimes W$, $t \neq 0$,

$$\begin{aligned}
D_t &:= \{d \otimes v \otimes w : d \in D\} \subseteqq D \otimes V \otimes W,\\
E_t &:= \{u \otimes e \otimes w : e \in E\} \subseteqq U \otimes E \otimes W,\\
F_t &:= \{u \otimes v \otimes f : f \in F\} \subseteqq U \otimes V \otimes F.
\end{aligned}$$

Clearly $\dim D_t = \dim D$, $\dim E_t = \dim E$, and $\dim F_t = \dim F$. Note that D_t is invariant under shifts of the triad t along U_t, i.e.,

$$D_t = D_{t'} \quad \Leftrightarrow \quad t' \in U_t; \tag{2.3a}$$

and analogously,

$$E_t = E_{t'} \quad \Leftrightarrow \quad t' \in V_t, \tag{2.3b}$$

$$F_t = F_{t'} \quad \Leftrightarrow \quad t' \in W_t. \tag{2.3c}$$

LEMMA 2.4. *The tangent space of $\mathbb{S}$ at (the triad) t is given by*

$$T_{t\mathbb{S}} = \begin{cases} U_t + V_t + W_t, & t \neq 0, \\ U \otimes V \otimes W, & t = 0. \end{cases}$$

Proof. Let $t \neq 0$. Then t is a regular point of $\mathbb{S}$, i.e., $\dim T_{t\mathbb{S}} = \dim \mathbb{S}$. Now $t \in U_t \subseteqq \mathbb{S}$ implies $U_t = T_{tU_t} \subseteqq T_{t\mathbb{S}}$, since U_t is a linear space. Analogously, $V_t \subseteqq T_{t\mathbb{S}}$, and $W_t \subseteqq T_{t\mathbb{S}}$. Thus

$$U_t + V_t + W_t \subseteqq T_{t\mathbb{S}} \qquad (t \neq 0).$$

Comparing dimensions on both sides of this inclusion, we get an equality.

$0 \in \operatorname{span}\{t\} \subseteqq \mathbb{S}$ implies $\operatorname{span}\{t\} = T_{0\operatorname{span}\{t\}} \subseteqq T_{0\mathbb{S}}$ for every triad t. Thus, by $\operatorname{span} \mathbb{S} = U \otimes V \otimes W$, $T_{0\mathbb{S}} = U \otimes V \otimes W$. ■

Equation (2.2) and Lemma 2.4 show that X_{r-1} is singular for X_r, i.e., $X_{r-1} \subseteqq \operatorname{Sing} X_r$ for $r < \underline{R}$ [just put $t_r = 0$ in (2.2)]. In order to exclude the singular zero tensor, in the sequel we put

$$S = S(U \otimes V \otimes W) = \mathbb{S} \setminus \{0\}.$$

The following lemma translates the well-known derivative criterion (Jacobian criterion) for algebraic dependence (cf. [22]) into the present situation.

LEMMA 2.5.

$$\exists \mathbf{t} \in S^r\colon \dim \sum_{\rho \leqslant r} T_{t_\rho S} \geqslant q \quad \Rightarrow \quad \dim X_r \geqslant q.$$

Proof. By Lemma 2.6 below, $\dim \sum_{\rho \leqslant r} T_{t_\rho S} \geqslant q$ constitutes an open condition in S^r. Therefore, if it holds for some $\mathbf{t} \in S^r$, it also holds for all $\mathbf{t}$ in some nonempty Zariski open subset of S^r. Thus we may assume that $\varphi_r(\mathbf{t})$ is a regular point of X_r. Then (cf. [17])

$$\dim X_r = \dim T_{\varphi_r(\mathbf{t})X_r} \geqslant \dim \operatorname{im} d_{\mathbf{t}\varphi_r} = \dim\Big(\sum T_{t_\rho S}\Big) \geqslant q. \qquad \blacksquare$$

REMARK. In Lemma 2.5 we have an equivalence if the extension $k(X_r) \subseteqq k(\mathbb{S}^r)$ of the function fields (induced by φ_r) is separably generated (cf. [22, Theorem 41, p. 127]). In particular this always holds in the case $\operatorname{char} k = 0$.

LEMMA (2.6). *Let $D^\rho \subseteqq U$, $E^\rho \subseteqq V$, $F^\rho \subseteqq W$ $(\rho \leqslant r)$, and $M \subseteqq U \otimes V \otimes W$ be subspaces. Then the mapping*

$$S^r \to \mathbb{Z}, \qquad \mathbf{t} \mapsto \dim\Bigg(\sum_{\rho \leqslant r} D^\rho_{t_\rho} + E^\rho_{t_\rho} + F^\rho_{t_\rho}\Bigg) \Big/ M$$

is Zariski lower semicontinuous, i.e., for every $q \in \mathbb{Z}$,

$$\left\{\mathbf{t} \in S^r : \dim\Bigg(\sum_{\rho \leqslant r} D^\rho_{t_\rho} + E^\rho_{t_\rho} + F^\rho_{t_\rho}\Bigg) \Big/ M \geqslant q\right\} \subseteqq S^r$$

is open.

Proof. Consider, for $d \in U$ and $u^* \in U^*$, the dual space of U, the linear mapping

$$A(d, u^*) : U \otimes V \otimes W \to U \otimes V \otimes W / M,$$

$$u \otimes v \otimes w \mapsto u^*(u) \cdot d \otimes v \otimes w + M.$$

Analogously, let $B(e, v^*)$ be defined by $u \otimes v \otimes w \mapsto v^*(v) \cdot u \otimes e \otimes w + M$ $(e \in V,\ v^* \in V^*)$ and $C(f, w^*)$ by $u \otimes v \otimes w \mapsto w^*(w) \cdot u \otimes v \otimes f + M$ $(f \in W,\ w^* \in W^*)$. Then

(2.7)

$$\dim\Big(\sum D^\rho_{t_\rho} + E^\rho_{t_\rho} + F^\rho_{t_\rho}\Big)/M < q$$

$$\Leftrightarrow \quad \dim \operatorname{span}\Big(\ldots, A(d, u^*)t_\rho, B(e, v^*)t_\rho, C(f, w^*)t_\rho, \ldots\Big) < q,$$

where in the latter list ρ is assumed to run from 1 to r, and for fixed ρ, the arguments d, e, f, u^*, v^*, and w^* are assumed to run through some bases of D^ρ, E^ρ, F^ρ, U^*, V^*, and W^*. By the determinantal criterion for linear dependence (2.7) can be expressed by algebraic equations, hence it is a closed condition, as desired. ∎

Now, how to show the existence required in Lemma 2.5? The following lemma indicates how to break up the tangent sum (2.2) into smaller disjoint pieces.

LEMMA 2.8. *Let $U = D^0 \oplus D^1$, $V = E^0 \oplus E^1$, and $W = F^0 \oplus F^1$. Then for $t \in S(D^0 \otimes E^0 \otimes F^0)$*

$$T_{tS(U \otimes V \otimes W)} = T_{tS(D^0 \otimes E^0 \otimes F^0)} \oplus D_t^1 \oplus E_t^1 \oplus F_t^1.$$

Proof.

$$\begin{aligned} T_{tS(U \otimes V \otimes W)} &= U_t + V_t + W_t = D_t^0 + D_t^1 + E_t^0 + E_t^1 + F_t^0 + F_t^1 \\ &= T_{tS(D^0 \otimes E^0 \otimes F^0)} \oplus D_t^1 \oplus E_t^1 \oplus F_t^1. \end{aligned}$$ ∎

In the following we need a language to express the notion that a linear space

$$T = \left(\sum_{\rho \leqslant q} T_{t_\rho S} + \sum_{q < \rho \leqslant r} \left[D_{t_\rho}^\rho + E_{t_\rho}^\rho + F_{t_\rho}^\rho \right] \right) \Big/ M \subseteqq U \otimes V \otimes W / M$$

has "maximal possible" dimension. Here M denotes a subspace of $U \otimes V \otimes W$.

DEFINITION 2.9. We call T total [with respect to the pieces $T_{t_\rho S}$ $(\rho \leqslant q)$ and $D_{t_\rho}^\rho$, $E_{t_\rho}^\rho$, $F_{t_\rho}^\rho$ $(q < \rho \leqslant r)$] if

$$\dim T = q \dim S + \sum_{q < \rho \leqslant r} (\dim D^\rho + \dim E^\rho + \dim F^\rho).$$

If L is a subspace of $U \otimes V \otimes W / M$, then we write

$$T \overset{*}{\subseteqq} L \quad \Leftrightarrow \quad T = L \text{ or } T \text{ total},$$

$$T \overset{*}{=} L \quad \Leftrightarrow \quad T = L \text{ and } T \text{ total}.$$

We rephrase the Jacobian criterion (2.5) in this language:

If $\Sigma_{\rho \leqslant r} T_{t_\rho S}$ is total for some $\mathbf{t} \in S^r$, then r is small;

if for every r, $\Sigma_{\rho \leqslant r} T_{t_\rho S} \overset{*}{\subseteqq} U \otimes V \otimes W$ for some $\mathbf{t} \in S^r$, then $U \otimes V \otimes W$ is good;

if $r = nml/(n+m+l-2)$ is an integer, and $\Sigma_{\rho \leqslant r} T_{t_\rho S} \overset{*}{=} U \otimes V \otimes W$ for some $\mathbf{t} \in S^r$, then $U \otimes V \otimes W$ is perfect.

3. UPPER BOUNDS ON THE TYPICAL RANK

In this section we apply the method of the previous paragraph. To this end we first give the following purely technical lemma.

LEMMA 3.1. *Let $U \otimes V \otimes W$ be of the shape (n, m, l). Then*

$$\sum_{\rho \leqslant p} T_{t_\rho S} + \sum_{p < \rho \leqslant q} U_{t_\rho} + \sum_{q < \rho \leqslant r} V_{t_\rho} \overset{*}{=} U \otimes V \otimes W \qquad \textit{almost everywhere in } S^r$$

in each of the following cases:

(1) $n = p = 1,\ q = r = (l-1)(m-1)+1$;
(2) $p = 0,\ l = l_1 + l_2,\ q = ml_1,\ r - q = nl_2$;
(3) $n = m = 2,\ p = 0,\ 2 \leqslant q \leqslant r - q,\ r = 2l$;
(4) $n = m = p = 2 \leqslant l = r$;
(5) $n = m = p = 3,\ q - 3 = r - q = l - 2 \geqslant 2$.

Proof. By Lemma 2.6 it suffices to show the existence. Let $t_\rho = u_\rho \otimes v_\rho \otimes w_\rho\ (\rho \leqslant r)$.

(1): Let $V = \operatorname{span}\{v_1\} \oplus E$ and $W = \operatorname{span}\{w_1\} \oplus F$. Apply Lemma 2.8 to t_1, and choose $(v_\rho \otimes w_\rho : 1 < \rho \leqslant r)$ as a basis of $E \otimes F$.

(2): Let $W = F^1 \oplus F^2$, where $\dim F^1 = l_1$ and $\dim F^2 = l_2$. Choose $(v_\rho \otimes w_\rho : \rho \leqslant q)$ as a basis of $V \otimes F^1$ and $(u_\rho \otimes w_\rho : q < \rho \leqslant r)$ as a basis of $U \otimes F^2$.

(3): By (2) we may assume that q is odd. Let $W = F^1 \oplus F^2$, where $\dim F^1 = 3$. We choose $w_\rho, w_{\rho+q} \in F^1$ $(\rho \leqslant 3)$ and $w_\rho \in F^2$ (else). It suffices to show that a.e. (almost everywhere) in $S(U \otimes V \otimes F^1)^6 \times S(U \otimes V \otimes F^2)^{r-6}$

$$\left(\sum_{\rho \leqslant 3} U_{t_\rho} + V_{t_{\rho+q}} \right) \oplus \left(\sum_{3 < \rho \leqslant q} U_{t_\rho} + \sum_{q+3 < \rho \leqslant r} V_{t_\rho} \right) = U \otimes V \otimes F^1 \oplus U \otimes V \otimes F^2.$$

By the openness principle (2.6) it suffices to prove the existence, and this may be done for both parts separately. The second part is settled by (2). The first part is left to the reader as an exercise. (Draw a $2\times2\times3$ illustration.)

(4): Let $W=F^1\oplus F^2\oplus F^3$, where $\dim F^1=2$, and $\dim F^2=q-2$. We choose $w_\rho\in F^1$ $(\rho\leqslant 2)$, $w_\rho\in F^2$ $(2<\rho\leqslant q)$, and $w_\rho\in F^3$ $(\rho>q)$. It suffices to show that a.e. in $S(U\otimes V\otimes F^1)^2\times S(U\otimes V\otimes F^2)^{q-2}\times S(U\otimes V\otimes F^3)^{r-q}$

$$\Big(\sum_{\rho\leqslant 2}T_{t_\rho S(U\otimes V\otimes F^1)}\Big)\oplus\Big(\sum_{2<\rho\leqslant q}U_{t_\rho}+\sum_{\rho\leqslant 2}F^2_{t_\rho}\Big)\oplus\Big(\sum_{\rho>q}V_{t_\rho}+\sum_{\rho\leqslant 2}F^3_{t_\rho}\Big)$$

$$=U\otimes V\otimes F^1\ \oplus\ U\otimes V\otimes F^2\ \oplus\ U\otimes V\otimes F^3.$$

(Use Lemma 2.8.) We give examples for the existence separately for each part. For the first part choose (u_1,u_2) as a basis of U, and apply case (1) twice. For the second and the third part apply case (2). (Change the roles of the corresponding spaces.)

(5): Let $W=F^1\oplus F^2$, where $\dim F^1=2$. We choose $w_\rho\in F^1$ $(\rho\leqslant 3)$ and $w_\rho\in F^2$ $(\rho>3)$. It suffices to show that a.e. in $S(U\otimes V\otimes F^1)^3\times S(U\otimes V\otimes F^2)^{2(l-2)}$

$$\Big(\sum_{\rho\leqslant 3}T_{t_\rho S(U\otimes V\otimes F^1)}\Big)\oplus\Big(\sum_{3<\rho\leqslant q}U_{t_\rho}+\sum_{q<\rho\leqslant r}V_{t_\rho}+\sum_{\rho\leqslant 3}F^2_{t_\rho}\Big)$$

$$=U\otimes V\otimes F^1\ \oplus\ U\otimes V\otimes F^2.$$

Again, we prove the existence for both parts separately. For the first part, choose (u_1,u_2,u_3) as a basis of U and apply case (1) three times. For the second part (call it T), we choose (u_1,u_2,u_3) as a basis of U and put $v_1=v_2$. Then, by (2), a. a. (almost always)

$$T\supseteqq\sum_{q<\rho\leqslant r}\operatorname{span}\{v_1\}_{t_\rho}+\sum_{\rho\leqslant 2}F^2_{t_\rho}=U\otimes\operatorname{span}\{v_1\}\otimes F^2,$$

and, using this, we obtain [again by (2)] that a.a.

$$T\supseteqq\Big(\sum_{q<\rho\leqslant r}\operatorname{span}\{v_1\}_{t_\rho}+\sum_{\rho\leqslant 2}F^2_{t_\rho}\Big)+\Big(\sum_{3<\rho\leqslant q}\operatorname{span}\{u_3\}_{t_\rho}+F^2_{t_3}\Big)$$

$$=U\otimes\operatorname{span}\{v_1\}\otimes F^2+\operatorname{span}\{u_3\}\otimes V\otimes F^2.$$

[Draw a $3\times3\times(l-2)$ illustration.] Thus it suffices to prove that a.a.

$$T\big/\big(U\otimes\operatorname{span}\{v_1\}\otimes F^2+\operatorname{span}\{u_3\}\otimes V\otimes F^2\big)$$
$$=U\otimes V\otimes F^2\big/\big(U\otimes\operatorname{span}\{v_1\}\otimes F^2+\operatorname{span}\{u_3\}\otimes V\otimes F^2\big).$$

But this is case (3) (up to isomorphy). ∎

In the subsequent proofs we will put the triads in the tangent sum (2.2) into different "positions" of the tensor-product space $U\otimes V\otimes W$ in order to split up the sum into disjoint parts, as listed in the previous Lemma. To avoid clumsy notation we will use in the sequel the following suggestive symbolism: Suppose $U=\bigoplus_{i\geqslant 0}D^i$, $V=\bigoplus_{j\geqslant 0}E^j$, and $W=\bigoplus_{k\geqslant 0}F^k$. Then we write

$$D^{i_1\cdots i_s}:=D^{i_1}\oplus\cdots\oplus D^{i_s},\tag{3.2}$$
$$D^A:=U\qquad(A=\text{all}),$$

and analogously $E^{j_1\cdots j_s},\dots,F^A$. With these position symbols we write

$$S_{IJK}:=S(D^I\otimes E^J\otimes F^K)\tag{3.2$'$}$$

[e.g., $S_{012A}=S((D^0\oplus D^1)\otimes E^2\otimes F^A)$]. Finally we attach to the "position" IJK a finite set, also denoted by IJK (meaning: $|IJK|$ many triads are chosen in position IJK, i.e., $\in S_{IJK}$). In this sense S_{IJK}^{IJK} denotes the $|IJK|$-fold Cartesian product of S_{IJK}.

THEOREM 3.3.

$$\underline{R}(n,m,l)=nml/(n+m+l-2)+O(l)\qquad(n\leqslant m\leqslant l).$$

Proof. Since $\underline{R}$ is monotonic in each argument, we may assume w.l.o.g. that m and l are even, say $m=2\mu$, $l=2\lambda$. Let $U\otimes V\otimes W$ be of the shape (n,m,l). We show for $r=l\lceil nm/(n+m+l-2)\rceil$

$$\exists t\in S^r:\quad\sum_{\rho\leqslant r}T_{t_\rho S}=U\otimes V\otimes W.$$

Then r is large, by Lemma 2.5. To this end let $t\in S^r$, $W=\bigoplus_{0\leqslant k<\lambda}F^k$, $[1,r]=\bigsqcup_{0\leqslant k<\lambda}AAk$ with $\dim F^k=2$, and $|AAk|=r/\lambda$ $(0\leqslant k<\lambda)$. We choose

$$t_\rho \in S_{AAk} \qquad (\rho \in AAk, \quad 0 \leqslant k < \lambda)$$

and obtain by Lemma 2.8 the decomposition

$$\begin{aligned}\sum_{\rho \leqslant r} T_{t_\rho S} &= \sum_{k=0}^{\lambda-1} \sum_{AAk} \Big(T_{t_\rho S_{AAk}} \oplus \bigoplus_{\gamma \neq k} F_{t_\rho}^{\gamma} \Big) \\ &= \bigoplus_{k=0}^{\lambda-1} \underbrace{\Big(\sum_{AAk} T_{t_\rho S_{AAk}} + \sum_{\sqcup_{\gamma \neq k} AA\gamma} F_{t_\rho}^{k} \Big)}_{(\text{call this } T_{AAk})}.\end{aligned}$$

[Here we use the convention that each sum just ranges over the spaces directly behind the Σ. The positions of the summands are immediate, e.g., $F_{t_\rho}^k \subseteqq D^A \otimes E^A \otimes F^k = U \otimes V \otimes F^k$ $(\rho \in AA\gamma)$.] By the openness principle (2.6) we may check for each k separately the existence of some $\mathbf{t} \in \prod_\gamma S_{AA\gamma}^{AA\gamma}$ such that $T_{AAk} = U \otimes V \otimes F^k$.

Now, let $V = \bigoplus_{0 \leqslant j < \mu} E^j$ with $\dim E^j = 2$ $(0 \leqslant j < \mu)$, and $AAk = \bigsqcup_{0 \leqslant j < \mu} Ajk$. (Observe that the new positions are specializations of the former ones.) Choosing

$$t_\rho \in S_{Ajk} \qquad (\rho \in Ajk, \quad 0 \leqslant j < \mu)$$

we get

$$T_{AAk} = \bigoplus_{0 \leqslant j < \mu} \underbrace{\Big(\sum_{Ajk} T_{t_\rho S_{Ajk}} + \sum_{\sqcup_{\beta \neq j} A\beta k} E_{t_\rho}^j + \sum_{\sqcup_{\gamma \neq k} Aj\gamma} F_{t_\rho}^k \Big)}_{(\text{call this } T_{Ajk})}.$$

Now we choose $|Ajk| \in \{0,2\}$ and $|\bigsqcup_{\gamma \neq k} Aj\gamma|$ suitably so that for all j $(0 \leqslant j < \mu)$

$$T_{Ajk} = U \otimes E^j \otimes F^k \qquad \text{a.e. in } \prod_{\beta,\gamma} S_{A\beta\gamma}^{A\beta\gamma}.$$

This is possible by Lemma 3.1(3), (4). [Check $|AAk| = r/\lambda \leqslant m$, and use

(2.3) to translate Lemma 3.1 into the present situation.] This proves the theorem. ∎

As a consequence of the previous proof and (2.1) we obtain the following:

COROLLARY 3.4. *(n, m, l) is perfect $(n \leqslant m \leqslant l)$, provided that m and l are even and $nm/(n+m+l-2)$ is an integer.*

REMARK. Refined versions of Theorem 3.3 can be found in [19, 16]. The proofs of these, however, are more involved.

4. CUBIC SHAPES

In this section we investigate the principal case $n = m = l$. The following proposition is a first refinement of Theorem 3.3 for cubic shapes.

PROPOSITION 4.1. *Let $U \otimes V \otimes W$ be of the shape (n, n, n), and $n \equiv 0 \bmod 3$, $n \neq 3$. Then*

(1) *$n^2/3$ is small;*
(2) *$n(n+1)/3$ is large.*

Proof. Again we use Lemma 2.5. Let $\mathrm{t} \in \mathrm{S}^r$ with r to be chosen later, $[1, r] = \bigsqcup_{0 \leqslant i, j < \nu} ijA$, where $\nu = n/3$. Let $U = \bigoplus_{0 \leqslant i < \nu} D^i$, $V = \bigoplus_{0 \leqslant j < \nu} E^j$ with $\dim D^i = \dim E^i = 3$ $(0 \leqslant i < \nu)$. Choosing

$$t_\rho \in \mathrm{S}_{ijA} \qquad (\rho \in ijA, \quad 0 \leqslant i, j < \nu)$$

we get

$$\sum_{\rho \leqslant r} T_{t_\rho S} = \bigoplus_{0 \leqslant i, j < \nu} \underbrace{\left(\sum_{ijA} T_{t_\rho S_{ijA}} + \sum_{\sqcup_{\alpha \neq i} \alpha jA} D^i_{t_\rho} + \sum_{\sqcup_{\beta \neq j} i\beta A} E^j_{t_\rho} \right)}_{(\text{call this } T_{ijA})}.$$

(1), $r = 3\nu^2$: We put $|ijA| = 3$ $(0 \leqslant i, j < \nu)$. Then Lemma 3.1(5) implies that for all i and j $(0 \leqslant i, j < \nu)$

$$T_{ijA} \overset{*}{\subseteqq} D^i \otimes E^j \otimes W \qquad \text{a.e. in } \prod_{\alpha, \beta} \mathrm{S}^{\alpha\beta A}_{\alpha\beta A}.$$

By $r < \lceil n^3/(3n-2) \rceil$ and the symmetry of all T_{ijA} (i.e., they are all total, since equality is excluded) this proves the first assertion. [Apply (2.3) and use the trivial fact that totality is stable under omission of some of the pieces.]

(2), $r = 3\nu^2 + \nu$: We put $|ijA| = 3 + \delta_{ij}$ $(0 \leqslant i, j < \nu)$. Then, similarly, Lemma 3.1(5) implies that for all i and j $(0 \leqslant i, j < \nu)$

$$T_{ijA} = D^i \otimes E^j \otimes W \qquad \text{a.e. in } \prod_{\alpha,\beta} S_{\alpha\beta A}^{\alpha\beta A},$$

as desired. ∎

Now, how to prove the precise result for cubic shapes, announced in the Introduction? Clearly, only in exceptional cases are the geometry of the "tangent three-leg" (Lemma 2.4) and the distribution technique of the previous proofs well matched. So we must arrange our analysis more carefully in details. The following proof of the main result of this paper partially follows lines suggested by V. Strassen.

First of all, we consider quasicubic shapes.

PROPOSITION 4.2 (Strassen [19, Corollary 3.10]). *We have*

$$\sum_{\rho \leqslant r} T_{t_\rho S} \overset{*}{=} U \otimes V \otimes W \qquad a.e. \text{ in } S^r$$

if $U \otimes V \otimes W$ is of the shape

$$(n, n, n+2), \quad n \not\equiv 2 \bmod 3, \quad \textit{and} \quad r = \frac{n(n+2)}{3},$$

or

$$(n, n, n-1), \quad n \equiv 0 \bmod 3, \quad \textit{and} \quad r = \frac{n^2}{3}.$$

These shapes are perfect.

This result can be proved similarly to the previous Proposition; see [19, 16] for a proof.

For what follows it is convenient to complete the position symbols (3.2) by

$$D^a := \bigoplus_{i>0} D^i \qquad (a = \text{all except } 0),$$

and analogously $E^a := \bigoplus_{j>0} E^j$, $F^a := \bigoplus_{k>0} F^k$.

The next lemma is an "angular" version of Lemma 3.1.

LEMMA 4.3. *Let $U \otimes V \otimes W$ be of the shape (n, m, l), $U = D^0 \oplus D^a$, $V = E^0 \oplus E^a$ with* $\dim D^0 = n_0$, $\dim D^a = n_a$, *and* $\dim E^0 = m_0$, $\dim E^a = m_a$. *Then*

$$\sum_{\rho=1}^{p_0} \left(D^0_{t_\rho} + E^0_{t_\rho}\right) + \sum_{\rho=p_0+1}^{p_1} D^0_{t_\rho} + \sum_{\rho=p_1+1}^{p_2} E^0_{t_\rho} + \sum_{\rho=p_2+1}^{p_3} W_{t_\rho} + \sum_{\rho=p_3+1}^{p_4} W_{t_\rho}$$

$$\stackrel{*}{=} D^0 \otimes V \otimes W + U \otimes E^0 \otimes W \qquad a.e.\ in\ S^{p_2}_{aaA} \times S^{p_3-p_2}_{0AA} \times S^{p_4-p_3}_{A0A}$$

in each of the following cases:

(1) $p_1, p_0 + p_2 - p_1 \equiv 0 \bmod l$, $p_1 \leqslant lm_a$, $p_0 + p_2 - p_1 \leqslant ln_a$, $p_3 - p_2 \leqslant (m - p_1/l)n_0$, $p_4 - p_3 \leqslant (n - (p_0 + p_2 - p_1)/l)m_0$, $p_4 - p_2 = (m - p_1/l)n_0 + [n - (p_0 + p_2 - p_1)/l]m_0 - n_0 m_0$;

(2) $n = m \geqslant 3$, $l = 2$, $n_0 = m_0 = 1$, $3 \leqslant p_0 = p_1 = p_2 \leqslant 2n - 3$, p_0 *odd*, $p_3 - p_2 = p_4 - p_3 = n - (p_0 + 1)/2$.

Proof. By the openness principle (Lemma 2.6) it suffices to prove the existence.

(1): Let $D^a = D^1 \oplus D^2$, $E^a = E^1 \oplus E^2$ with $\dim D^2 = (p_0 + p_2 - p_1)/l$, $\dim E^2 = p_1/l$. Let $[p_2 + 1, p_4] = 00A \sqcup 10A \sqcup 01A$. We choose

$$t_\rho \in S_{22A} \qquad (\rho \leqslant p_2),$$

$$t_\rho \in S_{ijA} \qquad (\rho \in ijA, \quad 0 \leqslant i, j \leqslant 1, \ ij = 0).$$

Then our sum splits into

$$\left(\sum_1^{p_1} D^0_{t_\rho}\right) \oplus \left(\sum_1^{p_0} E^0_{t_\rho} + \sum_{p_1+1}^{p_2} E^0_{t_\rho}\right) \oplus \bigoplus_{\substack{0 \leqslant i, j \leqslant 1 \\ ij = 0}} \left(\sum_{ijA} W_{t_\rho}\right),$$

which equals [by Lemma 3.1(2) applied to each summand]

$$\bigoplus_{\substack{0 \leqslant i, j \leqslant 2 \\ ij = 0}} D^i \otimes E^j \otimes W \qquad \text{a.e. in } \prod S_{\alpha\beta A}^{\alpha\beta A},$$

provided that the $|ijA|$ $(0 \leqslant i, j \leqslant 1,\ ij = 0)$ are suitably chosen. (Draw an illustration.)

(2) is left to the reader as an exercise. (First consider $n = 3$.) ■

Theorem 4.4. *(n, n, n) is good $(n \neq 3)$.*

Corollary 4.5. $\underline{R}(n, n, n) = \lceil n^3/(3n-2) \rceil$ $(n \neq 3)$.

The proof of Theorem 4.4 is composed of Proposition 4.2 and the following two lemmas.

Lemma 4.6. *Let $n \geqslant 5$, $U \otimes V \otimes W$ be of the shape (n, n, n), $U = D^0 \oplus D^a$, $V = E^0 \oplus E^a$, $D^a \otimes E^a \otimes W$ be of the shape $(n-2, n-2, n)$, and $\min\{n-1, 8\} \leqslant p \leqslant n$. Then*

$$\left(\sum_{\rho \leqslant p} T_{t_\rho S_{AAA}} + \sum_{p < \rho \leqslant q} \left(D_{t_\rho}^0 + E_{t_\rho}^0 \right) \right) \Big/ D^a \otimes E^a \otimes W$$

$$\overset{*}{\subseteqq} U \otimes V \otimes W / D^a \otimes E^a \otimes W \qquad a.e.\ in\ S_{AAA}^p \times S_{aaA}^{q-p}.$$

Lemma 4.7. *Let $U \otimes V \otimes W$ be of the shape (n, n, n), $U = D^0 \oplus D^a$, $V = E^0 \oplus E^a$, $W = F^0 \oplus F^a$, $D^a \otimes E^a \otimes F^a$ be of the shape $(n-1, n-1, n-2)$, and $6 \leqslant p \leqslant n-1$. Then*

$$\left(\sum_{\rho \leqslant p} T_{t_\rho S_{AAA}} + \sum_{p < \rho \leqslant q} \left(D_{t_\rho}^0 + E_{t_\rho}^0 + F_{t_\rho}^0 \right) \right) \Big/ D^a \otimes E^a \otimes F^a$$

$$\overset{*}{\subseteqq} U \otimes V \otimes W / D^a \otimes E^a \otimes F^a \qquad a.e.\ in\ S_{AAA}^p \times S_{aaa}^{q-p}.$$

Proof of Theorem 4.4. The cases $n \leqslant 2$ being trivial, let's look at $n \geqslant 3$. It turns out that $(3,3,3)$ [more generally $(n, n, 3)$, n odd] is not good, a fact independently discovered by V. Strassen and the author. (See [19, 16] for this case.) The case $n = 4$ can be settled by the method of the proof of Theorem 3.3; we skip over the straightforward proof and consider now the cases $n \geqslant 5$.

Let $U = D^0 \oplus D^a$, $V = E^0 \oplus E^a$, $W = F^0 \oplus F^a$, and

$U \otimes V \otimes W$ be of the shape (n,n,n) $(n \geqslant 5)$,

$D^a \otimes E^a \otimes F^a$ be of the shape $\begin{cases} (n-2, n-2, n), & n \not\equiv 1 \bmod 3, \\ (n-1, n-1, n-2), & n \equiv 1 \bmod 3. \end{cases}$

Let $\lfloor n^3/(3n-2) \rfloor \leqslant r \leqslant \lceil n^3/(3n-2) \rceil$, $\mathrm{t} \in \mathrm{S}^r_{\mathrm{AAA}}$, and $[1, r] = aaa \sqcup \mathrm{AAA}$ with

$$|aaa| = \begin{cases} (n-2)n/3, & n \not\equiv 1 \bmod 3, \\ (n-1)^2/3, & n \equiv 1 \bmod 3. \end{cases}$$

Restricting

$$t_\rho \in \mathrm{S}_{aaa} \qquad (\rho \in aaa),$$

we get the epimorphism

$$\sum_{\rho \leqslant r} T_{t_\rho \mathrm{S}_{\mathrm{AAA}}}$$

$$= \underbrace{\sum_{aaa} T_{t_\rho \mathrm{S}_{aaa}}}_{\subseteqq D^a \otimes E^a \otimes F^a} + \left(\sum_{\mathrm{AAA}} T_{t_\rho \mathrm{S}_{\mathrm{AAA}}} + \sum_{aaa} \left(D^0_{t_\rho} + E^0_{t_\rho} + F^0_{t_\rho} \right) \right)$$

$$\twoheadrightarrow \underbrace{\left(\sum_{aaa} T_{t_\rho \mathrm{S}_{aaa}} \right)}_{(\text{call this } T_{aaa})} \oplus \underbrace{\left(\sum_{\mathrm{AAA}} T_{t_\rho \mathrm{S}_{\mathrm{AAA}}} + \sum_{aaa} \left(D^0_{t_\rho} + E^0_{t_\rho} + F^0_{t_\rho} \right) \right) \Big/ D^a \otimes E^a \otimes F^a}_{(\text{call this } T_{\mathrm{AAA}})}$$

By Proposition 4.2 and Lemmas 4.6, 4.7

$$T_{aaa} \overset{*}{=} D^a \otimes E^a \otimes F^a,$$

$$T_{\mathrm{AAA}} \overset{*}{\subseteqq} U \otimes V \otimes W / D^a \otimes E^a \otimes F^a \qquad \text{a.e. in } \mathrm{S}^{aaa}_{aaa} \times \mathrm{S}^{\mathrm{AAA}}_{\mathrm{AAA}}.$$

Thus

$$\sum_{aaa \sqcup AAA} T_{t_\rho S_{AAA}} \overset{*}{\subseteqq} U \otimes V \otimes W \qquad \text{a.e. in } S_{aaa}^{aaa} \times S_{AAA}^{AAA},$$

proving the theorem [by (2.1)]. The restrictions of Lemmas 4.6, 4.7 are satisfied, except for $n = 7$. Here we only get that 18 is small. But X_{18} is a hypersurface, so 19 is large, by $\dim X_{18} < \dim X_{19}$. ■

An amusing consequence is the following "nonconstructive" proof for the existence of fast matrix multiplication algorithms: Let $\langle n, n, n\rangle$ denote the tensor associated to the $n \times n$ matrix multiplication. Then Corollary 4.5 implies

$$\underline{\mathrm{rk}}\, \langle 2,2,2 \rangle \leqslant \underline{R}(4,4,4) = 7.$$

Applying [18, Theorem 4.1], we get

$$\mathrm{rk}\langle n,n,n\rangle \leqslant O(n^{\log_2(7)+\varepsilon}) \qquad (\varepsilon > 0, \quad n \to \infty).$$

Despite of this simple upper bound, it seems to be quite difficult to decide whether $\underline{\mathrm{rk}}\langle 2,2,2\rangle$ equals 6 or 7 [19,15]. (For rank it is known to be 7 [10,21].)

Proof of Lemma 4.6. To simplify the presentation, let's assume that n and p are even. (The other cases can be settled as in the subsequent proof; a complete treatment is given in [16].) We prove the existence. Let $W = \bigoplus_{0 \leqslant k < \nu} F^k$, $[1, p] = \bigsqcup_{0 \leqslant k < \nu} AAk$, and $[p+1, q] = \bigsqcup_{0 \leqslant k < \nu} aak$, where $\dim F^k = 2$ and $|AAk| \in \{0,2\}$ $(0 \leqslant k < \nu = n/2)$. Choosing

$$t_\rho \in S_{iik} \qquad (\rho \in iik, \quad i \in \{a, A\}),$$

we get

$$\Bigg(\sum_{\rho \leqslant p} T_{t_\rho S_{AAA}} + \sum_{p < \rho \leqslant q} \left(D_{t_\rho}^0 + E_{t_\rho}^0 \right) \Bigg) \Big/ D^a \otimes E^a \otimes W$$

$$= \bigoplus_{0 \leqslant k < \nu} \underbrace{\Bigg(\sum_{AAk} T_{t_\rho S_{AAk}} + \sum_{aak} \left(D_{t_\rho}^0 + E_{t_\rho}^0 \right) + \sum_{\sqcup_{\gamma \neq k} AA\gamma} F_{t_\rho}^k \Bigg) \Big/ D^a \otimes E^a \otimes W}_{\text{(call this } T \text{)}}.$$

We claim that for all k $(0 \leqslant k < \nu)$

$$T_{AAk} \stackrel{*}{=} U \otimes V \otimes F^k / D^a \otimes E^a \otimes W \qquad \text{a.a.},$$

provided that

$$|aak| = \begin{cases} n-(p+2)/2 & \text{in the case} \quad |AAk| = 2, \\ 2n-(p+4)/2 & \text{in the case} \quad |AAk| = 0. \end{cases}$$

This proves the assertion. We treat the two cases separately.

Case $|AAk| = 2$. Let $D^a = D^1 \oplus D^2$, $E^a = E^1 \oplus E^2$ with $\dim D^1 = \dim E^1 = (p-2)/2$. Let $AAk = 00k$, $aak = 22k$, and $AA\gamma = 10\gamma \sqcup 01\gamma$ with $|10\gamma| = |01\gamma|$ $(\gamma \neq k)$. Choosing

$$t_\rho \in iik \qquad (\rho \in iik; \quad i \in \{0,2\}),$$

$$t_\rho \in ij\gamma \qquad (\rho \in ij\gamma; \quad \gamma \neq k, \ 0 \leqslant i, j \leqslant 1, \ i+j=1),$$

we obtain

$$\begin{aligned} T_{AAk} &= \Bigg[\Big(\sum_{00k} T_{t_\rho S_{00k}} \Big) \oplus \Big(\sum_{00k} D^1_{t_\rho} + \sum_{\sqcup_{\gamma \neq k} 10\gamma} F^k_{t_\rho} \Big) \oplus \Big(\sum_{00k} E^1_{t_\rho} + \sum_{\sqcup_{\gamma \neq k} 01\gamma} F^k_{t_\rho} \Big) \\ &\quad \oplus \Big(\sum_{00k} D^2_{t_\rho} + \sum_{22k} E^0_{t_\rho} \Big) \oplus \Big(\sum_{00k} E^2_{t_\rho} + \sum_{22k} D^0_{t_\rho} \Big) \Bigg] \Big/ D^a \otimes E^a \otimes W \\ &= \Bigg[\bigoplus_{\substack{0 \leqslant i,j \leqslant 2 \\ ij=0}} D^i \otimes E^j \otimes F^k \Bigg] \Big/ D^a \otimes E^a \otimes W \\ &= U \otimes V \otimes F^k / D^a \otimes E^a \otimes W \qquad \text{a.a.} \end{aligned}$$

implying the claim in the first case. [Apply Lemma 3.1(4), (2) to each summand; draw an illustration.]

Case $|AAk| = 0$ (appearing only for $p \geqslant 8$). If $|aak|$ is even, then the claim follows from Lemma 4.3(1) [use $|aak| \leqslant 2(n-2)$]. So we assume that $|aak|$ is odd. (Then $p \geqslant 10$.) Let $D^a = D^1 \oplus D^2$, $E^a = E^1 \oplus E^2$ with $\dim D^2 = \dim E^2 = 3$. Let $aak = 11k \sqcup 22k$ with $|22k| = 3$, and $\bigsqcup_{\gamma \neq k} AA\gamma = 02k \sqcup 20k \sqcup 0101\,k$ with $|02k| = |20k| = 3$ (the third position index being irrelevant, we may replace it by k). If we choose

$$t_\rho \in S_{iik} \qquad (\rho \in iik; \quad i \in \{1,2,01\}),$$

$$t_\rho \in S_{ijk} \qquad (\rho \in ijk; \quad i,j \in \{0,2\}, \ i+j=2),$$

we obtain

$$\begin{aligned}
T_{AAk} &= \Big(\sum_{11k}\big(D^0_{t_\rho}+E^0_{t_\rho}\big)+\sum_{0101k}F^k_{t_\rho}\Big)\Big/D^a\otimes E^a\otimes W\\
&\quad+\Big[\Big(\sum_{22k}D^0_{t_\rho}+\sum_{02k}F^k_{t_\rho}\Big)+\Big(\sum_{22k}E^0_{t_\rho}+\sum_{20k}F^k_{t_\rho}\Big)\Big]\Big/D^a\otimes E^a\otimes W\\
&= \big(D^{01}\otimes E^{01}\otimes F^k+D^0\otimes E^2\otimes F^k+D^2\otimes E^0\otimes F^k\big)\big/D^a\otimes E^a\otimes W\\
&= U\otimes V\otimes F^k/D^a\otimes E^a\otimes W \qquad \text{a.a.},
\end{aligned}$$

proving the claim in the second case. [Apply Lemma 4.3(1) using $|11k|\leqslant 2(n-5)$, and then Lemma 3.1(2) twice.] ■

Proof of Lemma 4.7. Here we assume that n is even and p is odd in order to simplify the presentation. (The other cases are similar; for a complete treatment we refer to [16].) Again, we prove the existence. Let

$$T=\Big(\sum_{\rho\leqslant p}T_{t_\rho S_{AAA}}+\sum_{p<\rho\leqslant q}\big(D^0_{t_\rho}+E^0_{t_\rho}+F^0_{t_\rho}\big)\Big).$$

We show

$$T/M \overset{*}{=} U\otimes V\otimes W/M \qquad \text{a.a.} \tag{4.8}$$

for $q-p=\frac{1}{4}[4n^2-5n+2-p(3n-2)]$, where $M=D^a\otimes E^a\otimes F^a$. First we treat the special case $p=n-1$. (Then $q-p=n^2/4$.) To this end let $n=2\nu$, and $F^a=F^1\oplus F^2$ with $\dim F^2=\nu$. Let $[p+1,q]=aa2$. We choose

$$t_\rho\in S_{aa2} \qquad (\rho\in aa2).$$

Then we have

$$\begin{aligned}
T/M &\supseteqq \Big(\sum_{\rho<n}F^2_{t_\rho}+\sum_{aa2}\big(D^0_{t_\rho}+E^0_{t_\rho}\big)\Big)\Big/M\\
&= U\otimes V\otimes F^2/M \qquad \text{a.e. in } S^{n-1}_{AAA}\times S^{\nu^2}_{aa2},
\end{aligned}$$

by Lemma 4.3. (The reader is advised to visualize the subsequent considerations by drawing suitable cubes.) So we show (4.8) for $M^1 = M + U \otimes V \otimes F^2$. We decompose $D^a = D^1 \oplus D^2 \oplus D^3$, and $E^a = E^1 \oplus E^2 \oplus E^3$, where $\dim D^1 = \nu$, $\dim D^2 = \nu - 2$, $\dim E^1 = \dim E^2 = \nu - 1$, and $\dim D^3 = \dim E^3 = 1$. Let $aa2 = 212 \sqcup 3a2 \sqcup a32$ with $|212| = \nu^2 - (3\nu - 2)$, $|3a2| = \lceil \frac{3}{2}\nu - 1 \rceil$, and $|a32| = \lfloor \frac{3}{2}\nu - 1 \rfloor$. Choosing

$$t_\rho \in S_{ijk} \qquad (\rho \in ijk, \quad ijk \in \{212, 3a2, a32\})$$

we get

$$\begin{aligned} T/M^1 &\supseteqq \left(\sum_{\rho=1}^{2\nu-1} \left(D^3_{t_\rho} + E^3_{t_\rho} \right) + \sum_{3a2 \sqcup a32} F^0_{t_\rho} \right) \Big/ M^1 \\ &= \left(D^3 \otimes V \otimes W + U \otimes E^3 \otimes W \right) / M^1 \\ &\qquad \text{a.e. in } S^{2\nu-1}_{AAA} \times S^{212}_{212} \times S^{3a2}_{3a2} \times S^{a32}_{a32}. \end{aligned}$$

[Distribute appropriately, and use Lemma 3.1(2) in $D^3 \otimes E^0 \otimes F^{01}$ and in $D^0 \otimes E^3 \otimes F^{01}$, and Lemma 4.3 in $D^3 \otimes E^a \otimes F^0 + D^a \otimes E^3 \otimes F^0$.] So we show (4.8) for $M^2 = M^1 + D^3 \otimes V \otimes W + U \otimes E^3 \otimes W$. Let $[1, 2\nu - 1] = A01A \sqcup A2A$ with $|A01A| = |A2A| + 1 = \nu$. We choose

$$t_\rho \in S_{AjA} \qquad (\rho \in AjA, \quad j \in \{01, 2\})$$

and obtain

$$\begin{aligned} T/M^2 &\supseteqq \left(\sum_{A2A} T_{t_\rho S_{A2A}} + \sum_{A01A} E^2_{t_\rho} \right) \Big/ M^2 \\ &= U \otimes E^2 \otimes W / M^2 \qquad \text{a.e. in } S^{\nu}_{A01A} \times S^{\nu-1}_{A2A} \times S^{212}_{212} \times S^{3a2}_{3a2} \times S^{a32}_{a32}. \end{aligned}$$

[Choose $t_\rho \in S_{01220}$ ($\rho \in A2A$), and use Lemma 3.1(2) in $D^0 \otimes E^2 \otimes F^1$, and Lemma 3.1(1) $\nu - 1$ times ($= \dim E^2$ times) in $D^{012} \otimes E^2 \otimes F^0$.] So we show (4.8) for $M^3 = M^2 + U \otimes E^2 \otimes W$. Next we realize

$$T/M^3 \supseteqq \left(\sum_{A01A} D^2_{t_\rho} + \sum_{212} F^0_{t_\rho} \right) \Big/ M^3 = D^2 \otimes E^{01} \otimes W / M^3 \qquad \text{a.a.}$$

[Use Lemma 3.1(2) in $D^2 \otimes E^0 \otimes F^{01}$ and in $D^2 \otimes E^1 \otimes F^0$.] So we show (4.8) for $M^4 = M^3 + D^2 \otimes E^{01} \otimes W$. Let $A01A = 1010$. Choosing

$$t_\rho \in S_{1010} \qquad (\rho \in 1010),$$

we get

$$T/M^4 \supseteqq \left(\sum_{1010} T_{t_\rho S_{1010}} \right) \Big/ M^4 = D^1 \otimes E^{01} \otimes F^0 / M^4$$
$$\text{a.a. in } S^\nu_{1010} \times S^{\nu-1}_{A2A} \times S^{212}_{212} \times S^{3a2}_{3a2} \times S^{a32}_{a32}.$$

[Use Lemma 3.1(1) ν times (= dim D^1 times) in $D^1 \otimes E^{01} \otimes F^0$.] So we show (4.8) for $M^5 = M^4 + D^1 \otimes E^{01} \otimes F^0$. Since

$$T/M^5 \supseteqq \left(\sum_{1010} F^1_{t_\rho} \right) \Big/ M^5 = D^1 \otimes E^0 \otimes F^1 / M^5 \qquad \text{a.a.}$$

[by Lemma 3.1(2)], it suffices to show (4.8) for $M^6 = M^5 + D^1 \otimes E^0 \otimes F^1$. But

$$T/M^6 \supseteqq \left(\sum_{1010} D^0_{t_\rho} + \sum_{A2A} E^{01}_{t_\rho} \right) \Big/ M^6 = U \otimes V \otimes W / M^6 \qquad \text{a.a.}$$

[Use Lemma 3.1(2) in $D^0 \otimes E^{01} \otimes F^{01}$.] This proves (4.8) in the special case.

Now, we expand the case just proved ($p = n - 1$) to the general one. Let $D^a = D^1 \oplus D^2$, $E^a = E^1 \oplus E^2$, and $F^a = F^1 \oplus F^2$, where $\dim D^2 = \dim E^2 = \dim F^2 = n - p - 1$. Let $[p+1, q] = aa1 \sqcup aa2 \sqcup 222$ with $|aa2| = \frac{1}{2}(n-p-1)(p+1)$, $|222| = (n-p-1)^2$, and $[1, p] = AA01$. Choosing

$$t_\rho \in S_{ijk} \qquad (\rho \in ijk, \quad ijk \in \{aa1, aa2, 222, AA01\}),$$

we get

$$T/M = \Bigg[\underbrace{\sum_{AA01} T_{t_\rho S_{AA01}} + \sum_{aa1} \left(D^0_{t_\rho} + E^0_{t_\rho} \right) + \sum_{aa1 \sqcup aa2} F^0_{t_\rho}}_{(\text{call this } T_{AA01})}$$
$$\underbrace{+ \sum_{AA01} F^2_{t_\rho} + \sum_{aa2 \sqcup 222} \left(D^0_{t_\rho} + E^0_{t_\rho} \right) + \sum_{222} F^0_{t_\rho} \Bigg] \Big/ M,}$$
$$= (U \otimes V \otimes F^2 + D^2 \otimes E^2 \otimes F^0)/M$$
$$\text{a.e. in } S^p_{AA01} \times S^{aa1}_{aa1} \times S^{aa2}_{aa2} \times S^{222}_{222}$$

[by Lemma 4.3 and Lemma 3.1(2)]. So it suffices to prove that for $M^1 = M + U\otimes V\otimes F^2 + D^2\otimes E^2\otimes F^0$,

$$T_{AA01}/M^1 \overset{*}{=} U\otimes V\otimes W/M^1 \qquad \text{a.a.} \tag{4.9}$$

We continue the specialization to break up T_{AA01}/M^1. Let $AA01 = 010101$, $aa2 = 122 \sqcup 212$, and $aa1 = 111 \sqcup 121 \sqcup 211 \sqcup a21 \sqcup 2a1$. We choose

$$t_\rho \in S_{ijk} \qquad (\rho \in ijk, \quad ijk \in \{010101, 122, 212, 111, 121, 211, a21, a21\})$$

and get the epimorphism

$$
\begin{aligned}
T_{AA01}/M^1 \twoheadrightarrow \\
&\underbrace{\Big(\sum_{010101} T_{t_\rho S_{010101}} + \sum_{111\sqcup 211} D^0_{t_\rho} + \sum_{111\sqcup 121} E^0_{t_\rho} + \sum_{111} F^0_{t_\rho}\Big)\Big/M^1}_{\subseteq D^{01}\otimes E^{01}\otimes F^{01}/M^1} \\
&\oplus\Big(\sum_{010101} D^2_{t_\rho} + \sum_{a21\sqcup 2a1\sqcup 211} E^0_{t_\rho} + \sum_{2a1\sqcup 212\sqcup 211} F^0_{t_\rho} \\
&\quad + \sum_{010101} E^2_{t_\rho} + \sum_{a21\sqcup 2a1\sqcup 121} D^0_{t_\rho} + \sum_{a21\sqcup 122\sqcup 121} F^0_{t_\rho}\Big)\Big/ \\
&\quad (M^1 + D^{01}\otimes E^{01}\otimes F^{01}) \\
\cong &\underbrace{\Big(\sum_{010101} T_{t_\rho S_{010101}} + \sum_{111\sqcup 211} D^0_{t_\rho} + \sum_{111\sqcup 121} E^0_{t_\rho} + \sum_{111} F^0_{t\rho}\Big)}_{(\text{call this } T_{010101})}\Big/ D^1\otimes E^1\otimes F^1 \\
&\oplus \underbrace{\Big(\sum_{010101} D^2_{t_\rho} + \sum_{a21\sqcup 2a1\sqcup 211} E^0_{t_\rho} + \sum_{2a1\sqcup 212\sqcup 211} F^0_{t\rho}\Big)}_{(\text{call this } T_{20101})}\Big/ \\
&\quad (M^1 + D^{01}\otimes E^{01}\otimes F^{01}) \\
&\oplus \underbrace{\Big(\sum_{010101} E^2_{t_\rho} + \sum_{a21\sqcup 2a1\sqcup 121} D^0_{t_\rho} + \sum_{a21\sqcup 122\sqcup 121} F^0_{t_\rho}\Big)}_{(\text{call this } T_{01201})}\Big/ \\
&\quad (M^1 + D^{01}\otimes E^{01}\otimes F^{01}).
\end{aligned}
$$

Now, our aim is to achieve that a.e. in $S^p_{010101} \times \prod S^{\alpha\beta\gamma}_{\alpha\beta\gamma}$

$$(4.10)\qquad T_{010101}/D^1\otimes E^1\otimes F^1 \overset{*}{=} D^{01}\otimes E^{01}\otimes F^{01}/D^1\otimes E^1\otimes F^1,$$

(4.11)

$$T_{20101}/(M^1 + D^{01}\otimes E^{01}\otimes F^{01}) \overset{*}{=} D^2\otimes E^{01}\otimes F^{01}/(M^1 + D^{01}\otimes E^{01}\otimes F^{01}),$$

(4.12)

$$T_{01201}/(M^1 + D^{01}\otimes E^{01}\otimes F^{01}) \overset{*}{=} D^{01}\otimes E^2\otimes F^{01}/(M^1 + D^{01}\otimes E^{01}\otimes F^{01}),$$

proving (4.9). Note that the dimensions of the spaces on the right-hand side in (4.11), (4.12) are $(n-p-1)[3(p+1)-2]$. We distinguish two cases.

Case $\frac{1}{2}(n-p-1)(3p+1) \equiv 0 \bmod 4$. We choose $|122| = |212| = \frac{1}{2}|aa2|$, $|a21| = |2a1| = \frac{1}{8}(n-p-1)(3p+1)$, $|111| = (p+1)^2/4$, and $|121| = |211| = 0$. Then (4.10), (4.11), and (4.12) are achieved. [For (4.11) distribute appropriately in $n-p-1$ ($=\dim D^2$) "angles," and apply to each Lemma 4.3(1), analogously for (4.12). Equation (4.10) is the first proved case, $p = n-1$.]

Case $\frac{1}{2}(n-p-1)(3p+1) \equiv 2 \bmod 4$. We choose $|122| = |212| = \frac{1}{2}|aa2|$, $|a21| = |2a1| = \frac{1}{8}(n-p-1)(3p+1) - \frac{3}{2}$, $|111| = (p+1)^2/4 - 1$, and $|121| = |211| = 2$. Then, as above, (4.11) and (4.12) are achieved. Equation (4.10) needs some extra considerations. Assume first $|121| = |211| = 0$. Then $T_{010101}/D^1\otimes E^1\otimes F^1$ is a.a. total. Choose some $t \in S_{121}$. We assert that

$$\left(T_{010101} + E^0_t\right)/D^1\otimes E^1\otimes F^1 \text{ is a.a. total.}$$

Assume otherwise. Then

$$D^1\otimes E^0\otimes F^1/D^1\otimes E^1\otimes F^1 \subseteqq T_{010101}/D^1\otimes E^1\otimes F^1 \qquad \text{a.a.}$$

But then

$$\begin{aligned} T_{010101}/D^1\otimes E^1\otimes F^1 &\supseteqq \Big(\sum_{010101} F^1_{t_p} + \sum_{111} D^0_{t_p} + D^1\otimes E^0\otimes F^1\Big)/D^1\otimes E^1\otimes F^1 \\ &= D^{01}\otimes E^{01}\otimes F^1/D^1\otimes E^1\otimes F^1 \qquad \text{a.a.} \end{aligned}$$

[Use Lemma 3.1(2) in $D^0\otimes E^{01}\otimes F^1$.] Now, use Lemma 3.1(1), (2) (in

$U \otimes V \otimes F^0$) to conclude that $T_{010101}/D^1 \otimes E^1 \otimes F^1 = D^{01} \otimes E^{01} \otimes F^{01}/D^1 \otimes E^1 \otimes F^1$ a.a. Contradiction (count parameters). Successive application of this argument yields (4.10) in this case, too. ■

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