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AN ALGEBRAIC SOLUTION FOR A CLASS OF SUBJECTIVE METRICS MODELS

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It is shown that an obvious generalization of the subjective metrics model by Bloxom, Horan, Carroll and Chang has a very simple algebraic solution which was previously considered by Meredith in a different context. This solution is readily adapted to the special case treated by Bloxom, Horan, Carroll and Chang. In addition to being very simple, this algebraic solution also permits testing the constraints of these models explicitly. A numerical example is given.

1. *Introduction*

Among the presently available metric scaling models which allow for individual differences the points of view model by Tucker and Messick [1963] is probably the most widely known. More recently, Bloxom [1968], Horan [1969], and Carroll and Chang [1970] have developed an alternative which has a certain geometrical and intuitive appeal which is not immediately apparent in the older "points of view" model [Ross, 1966].

The individual difference model proposed by Horan, Bloxom, Carroll and Chang, in contrast, is geometrically quite straightforward. The gist of it is that, given the Euclidean coordinates of a set of stimuli, each subject is presumed to utilize his own ("elliptical") distance function for assessing the distances between pairs of stimuli; thus if

$$\alpha_i' = (a_{i1}, \cdots, a_{im}) \quad \text{and} \quad \alpha_j' = (a_{j1}, \cdots, a_{jm}) \tag{1.1}$$

are the coordinate vectors of two stimuli in the postulated latent space and the k'th subject is asked to report the distance between stimulus i and stimulus j, then he will reply with

$$d_k(i, j) = \sqrt{(\alpha_i - \alpha_j)' D_k^2 (\alpha_i - \alpha_j)}. \tag{1.2}$$

It is well known that scalar functions on vector pairs such as (1.2) satisfy all three distance axioms, as indeed do the slightly more general functions

$$d_k(i, j) = \sqrt{(\alpha_i - \alpha_j)' C_k (\alpha_i - \alpha_j)} \tag{1.3}$$

provided C_k is positive definite. Such distance functions are sometimes called "elliptical" metrics [*e.g.*, Pease, 1965, p. 219], and since the model in effect states that each subject utilizes his own metric, it is not farfetched to call such a scaling model a "model of subjective metrics."

The computational solutions proposed to date are either incomplete [Horan, 1969] or involve iteration [Bloxom, 1968; Carroll & Chang, 1970]. But it turns out that an algebraic solution of this model is not only possible, but surprisingly simple. In presenting this solution, we do not necessarily recommend its use in preference over presently available iterative algorithms. Algebraic solutions sometimes have a tendency to become unstable in the fallible case, and it is therefore often safer to replace them in actual applications by algorithms which have well understood optimality properties, such as least squares procedures, even at the cost of iteration. Nonetheless, the fact that an algebraic solution of the subjective metrics model exists ought to be of some interest in itself. Moreover, such algebraic solutions often offer an opportunity to study the properties of the model in a new light. For example, we will find that our present solution can be used to render the testable constraints of the subjective metrics models more explicit, *i.e.*, it can be used in actual applications to fallible data to test whether the subjective metrics model is appropriate.

2. The Constraints of the Subjective Metrics Model

A statement of the basic assumption of the subjective metrics model which is equivalent to (1.2) is

$$A_k = AD_k \qquad k = 1, N \tag{2.1}$$

where $A(p \times m)$ gives the Euclidean coordinates of the p stimuli in a space of m dimensions relative to some arbitrarily chosen origin (usually the centroid of the p points) and D_k is a positive definite diagonal matrix which defines the metric D_k^2 of the k'th subject. It is clear that (1.2) and (2.1) imply each other because the natural norm on the p rows α'_{ik} of A_k corresponds to the weighted norm on the rows α'_i of A, *i.e.*,

$$\alpha'_{ik}\alpha_{ik} = \alpha'_i D_k^2 \alpha_i \tag{2.2}$$

with an analogous relation among the two distance functions defined on the difference vectors in (1.2).

The proponents of this particular model all emphasize the important fact that use of such weighted norms eliminates the rotational indeterminacy, *i.e.*, as long as $D_k^2 \neq I_m$ (and has distinct diagonal elements) the only isometries left in (1.2) are translations which we assume to have taken care of ahead of time. The interpretation of this observation requires some care. The coordinate matrices A_k are, of course, rotationally invariant w.r.t. the observed matrices of scalar products

$$B_k = -(I - JJ'/p)\Delta_k^{(2)}(I - JJ'/p)/2 \tag{2.3}$$

($\Delta_k^{(2)}$ a matrix of squared distances, J a vector of ones. This would put the origin at the centroid). What cannot be rotated is A because any nontrivial rotation of A would change the numerical values of the norms in (2.2) and thus of the metrics in (1.2) if $D_k^2 \neq I_m$. This lack of rotational invariance of A can be interpreted as an asset, because it serves to identify the underlying common matrix of coordinates up to a diagonal matrix D

$$A_k = (AD)(D^{-1}D_k) \tag{2.4}$$

once the origin has been chosen, rather than up to a rotation, as would be the case if all the metrics were Euclidean. We feel, however, that some advocates of this model [*e.g.* Carroll and Chang, 1970] stretch this point a bit when they try to attach, on this premise alone, psychological significance to the directions of the columns of A, so identified. The problem is the same as in principal component analysis. The fact that principal components are rendered unique by a mathematically convenient constraint does not necessarily mean that they are also psychologically meaningful or useful.

Eq. (2.1) reveals the basic constraints implied by such a model: (i) all N matrices A_k are assumed to be in the same column space, that of A and (ii) D_k is diagonal. If we were to discard (ii) but retain (i), we would be dealing with the metrics in (1.3). The stronger diagonality constraint of the present model implies that the subject-specific weights in D_k or D_k^2 are defined on a ratio scale (Eq. (2.4)).

3. An Algebraic Solution of a Generalized Subjective Metrics Model

One obtains an obvious generalization of the subjective metrics model considered by Bloxom, Horan, Carroll and Chang upon replacing the specialized elliptical metric (1.2) by the general elliptical metric (1.3) and, thus, the defining equation (2.1) by

$$A_k = AT_k \qquad k = 1, N. \tag{3.1}$$

T_k is an arbitrary Gram factor of a positive definite matrix

$$C_k = T_kT_k' \qquad k = 1, N \tag{3.2}$$

which defines the metric C_k of the k'th subject as in (1.3), and A is defined as before. This more general model, which, of course, contains that considered by Bloxom, Horan, Carroll and Chang as a special case, has a very simple solution which has been previously studied by Meredith [1964] in an entirely different context.

An important difference between (2.1) and (3.1) is that A in (3.1) is defined only up to a nonsingular transformation on the right. This indeterminacy can be removed, in part, by imposing the constraint

$$\sum_k^N C_k/N = I_m \tag{3.3}$$

on the C_k . This constraint can always be imposed. Thus, if the average of $C_k^*(k = 1, N)$ is $C^* = TT'$, then $C_k = T^{-1}C_k^*(T')^{-1}$, $(k = 1, N)$ will satisfy (3.3) and the C_k , together with $A = A^*T$, will satisfy $A^*C_k^*A^{*\prime} = AC_kA'$. Note that (3.3) does not entirely resolve the indeterminacy of A and C_k , rather they are now determined up to a joint rotation.

The average matrix of scalar products B_k (2.3) can then be factored for A or AS ($SS' = S'S = I$) as a consequence of the constraint:

$$B = \sum_k^N B_k/N = \sum_k^N AC_kA'/N = AA' = (AS)(S'A'), \tag{3.4}$$

$$SS' = S'S = I_m .$$

The arbitrary rotation S can be removed in any manner desired in view of the rotational indeterminacy of A. For example, one might subject AS to an "orthogonal rotation" to simple structure to define A.

Having decided on A, one finds the associated C_k from (2.3)

$$C_k = (A'A)^{-1}A'B_kA(A'A)^{-1}, \qquad k = 1, N. \tag{3.5}$$

That this is a solution follows from Eqs. (2.3), (2.4), which imply

$$A(A'A)^{-1}A'B_kA(A'A)^{-1}A' = B_k$$

so that $AC_k^*A' = B_k$ if C_k^* is chosen as in (3.5). That this solution is, in fact, unique follows from the contradiction obtained upon assuming the contrary: If $C_k^* \neq C_k^{**}$ but $AC_k^*A' = AC_k^{**}A' = B_k$, then $A(C_k^* - C_k^{**})A' = \phi$ or $C_k^* = C_k^{**}$ since A, being of full column rank, has a left inverse.

This completes the algebraic solution of the generalized subjective metrics model. Carroll and Chang [1970] sketch another solution the exact status of which is unclear. They do not discuss the consistency issue. Unable to adapt it to the special case of the subjective metrics model they are actually interested in, they eventually resort to iteration and ignore the constraints of both models.

As mentioned earlier, the only substantive constraint in this model is that all matrices A_k are in the same column space, which is precisely the situation considered by Meredith in (Eq. (2), loc. cit.). This constraint is easily tested with the help of the orthogonal projector

$$P_A = A(A'A)^{-1}A' \tag{3.6}$$

for that column space in the fallible case. Thus, one might compute

$$E_k = (I - P_A)A_k^* \quad \text{and} \quad e_k = \operatorname{tr} E_k'E_k \qquad k = 1, N \tag{3.7}$$

for all (arbitrary) Gram factors A_k^* of B_k and then decide on the basis of e_k whether the assumption (i), above, is indeed warranted for the data on hand. In some instances it may be tenable for some subjects but not for others, which then could be dropped from the further analysis. To ensure that assumption (i) is met from this point on for all retained subjects, one might continue the analysis with

$$\hat{A}_k^* = P_A A_k^* \qquad k = 1, N \tag{3.8}$$

which is the least squares approximation to A_k^* in the column space of A, where A is based on the group average.

4. An Algebraic Solution of the Horan, Bloxom, Carroll and Chang Model

We now impose the second constraint of the subjective metrics model, so that

$$B_k = AD_k^2A' \qquad k = 1, N, \tag{4.1}$$

by (2.1), where D_k^2 = diagonal. To remove the indeterminacy (2.4) in part we again impose the constraint (3.3) which now determines A and D_k^2 up to a joint permutation and, perhaps, some reflections. Given N matrices B_k from (2.3), one can find a matrix A^* and N matrices C_k^* from (3.4) and (3.5) by ignoring the second constraint temporarily. This determines A^* and the N C_k^* up to a rotation S which, in view of (4.1), can be chosen so as to diagonalize all C_k^* simultaneously, to give A and the D_k^2 up to a joint permutation and the reflections. (Since the reflections do not affect the final result in (4.7), we shall find it convenient to ignore them altogether. Thus, when we use the term "permutation Q_k," we actually mean "a permutation matrix up to reflections.")

The latent roots and vectors of any one C_k^* are determined up to a joint permutation Q_k

$$C_k^* = (V_kQ_k)(Q_k'D_k^2Q_k)(Q_k'V_k') = V_k^*D_k^{*2}V_k^{*\prime} \qquad k = 1, N \tag{4.2}$$

if we assume, for convenience, that all roots are positive and distinct. (Complications which arise in the presence of multiple and/or zero roots will be briefly discussed in Sec. 5.) Since any one of these permutations can be fixed arbitrarily, we set

$$Q_1 = I. \tag{4.3}$$

Then, from (3.1) and (4.1)

$$B_1 = A^*C_1^*A^{*\prime} = A^*V_1D_1^2V_1'A^{*\prime} = AD_1^2A' \tag{4.4}$$

if we take

$$A = A^*V_1, \quad \text{i.e.,} \quad S = V_1. \tag{4.5}$$

Now let Q_k be that particular permutation in (4.2) which relates the k'th set of vectors V_k^* to V_1

$$V_k^* = V_1Q_k \quad \text{or} \quad Q_k = V_1'V_k^* \qquad k = 2, N. \tag{4.6}$$

In this case

$$
\begin{aligned}
B_k &= A^*C_k^*A^{*\prime} \\
&= AV_1'C_k^*V_1A' = AV_1'(V_1Q_k)(Q_k'D_k^2Q_k)(Q_k'V_1')V_1A' \qquad (4.7)\\
&= AD_k^2A'
\end{aligned}
$$

as demanded by the stronger model, and

$$
V_1'C_k^*V_1 = V_1'(V_1Q_k)(Q_k'D_k^2Q_k)(Q_k'V_1')V_1 = D_k^2 . \qquad (4.8)
$$

This means that V_1 , the set of eigenvectors of C_1^* for the arbitrarily fixed first subject, can be used to eliminate the permutations Q_k $(k = 2, N)$ so as to obtain the diagonal elements of D_k^2 $(k = 2, N)$ in the correct order relative to that in D_1^2 .

Eq. (4.8), in effect, renders the stronger condition (ii) of the Horan, Bloxom, Carroll and Chang model explicit, and thus testable. We thus have an answer to the representation problem for this model:

Theorem: The subjective metrics model proposed by Horan, Bloxom, Carroll and Chang has a solution if and only if

(i) All matrices B_k in (2.3) $(k = 1, N)$ are Gramian and of same rank m,
(ii) $(I - P_A)B_k = \phi$ for $k = 1, N$
where A is defined as in (3.4) and P_A as in (3.6); and
(iii) $V_1'C_k^*V_1$ = diagonal for $k = 2, N$
where C_k^* is defined as in (3.5) and V_1 is the set of (orthogonalized, if necessary) unit length eigenvectors of any given C_1^* .

Equivalently, one might also consider computing the permutation matrices Q_k in (4.6) and use them to reorder the diagonal elements in $V_1'C_k^*V_1$ $(k = 2, N)$ as we had suggested in an earlier draft of this paper. The present, more direct approach was suggested by Dr. Bloxom. We thus have an explicit check on both defining assumptions of the model, (i) and (ii), in (3.7) and (4.8), respectively. For a numerical illustration of the algebra, see Tables 1 and 2.

5. Discussion

In the error-free case these solutions are exact and unique under the stated assumptions and constraints. We now turn to consider briefly some of the complications which may arise in the case of multiple roots, zero roots and with fallible data.

If all roots are positive (negative roots are ruled out by (2.1) and (3.1) in the exact case and by the construction of the C_k in (3.5) even in the fallible case) but some are repeated within one or more subjects, then certain rotational indeterminacies within subspaces reappear which may or may not destroy the identifiability of A. This is no problem for the more general model, because A is only determined up to a rotation anyway. If in the stronger model at least one of the C_k^* (in (4.2), obtained from (3.5)) has a complete set of distinct positive roots, then this matrix can be used to identify V_1 and, thus, A. The rotational indeterminacies within the subspaces associated with repeated roots in some of the other C_k^* would be of no consequence. Although some of the Q_k^* would no longer be permutation matrices, but

TABLE 1

Numerical Illustration: Exact Case

Common coordinate matrix A

$$A = \begin{bmatrix} 1.0 & 2.0 \\ 2.0 & 3.0 \\ 0.0 & 1.0 \\ 2.0 & 0.0 \end{bmatrix}$$

Subjective metrics D_k^2

$$D_1^2 = (1.0 \quad 1.5)$$
$$D_2^2 = (0.7 \quad 0.5)$$
$$D_3^2 = (1.3 \quad 1.0)$$

Scalar product matrices B_k

$$B_1 = \begin{bmatrix} 7.0 & 11.0 & 3.0 & 2.0 \\ & 17.0 & 4.5 & 4.5 \\ & & 1.5 & 0.0 \\ \text{sym.} & & & 4.0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 2.7 & 4.4 & 1.0 & 1.4 \\ & 7.3 & 1.5 & 2.8 \\ & & 0.5 & 0.0 \\ \text{sym.} & & & 2.8 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 5.3 & 8.6 & 2.0 & 2.6 \\ & 14.2 & 3.0 & 5.2 \\ & & 1.0 & 0.0 \\ \text{sym.} & & & 5.2 \end{bmatrix}$$

Gram factors A_k^*

$$A_1^* = \begin{bmatrix} 2.63 & -.32 \\ 4.18 & -.05 \\ 1.07 & -.60 \\ .98 & 1.74 \end{bmatrix}$$

$$A_2^* = \begin{bmatrix} 1.61 & -.35 \\ 2.69 & -.22 \\ .52 & -.48 \\ 1.14 & 1.23 \end{bmatrix}$$

$$A_3^* = \begin{bmatrix} 2.25 & -.47 \\ 3.76 & -.27 \\ .75 & -.66 \\ 1.51 & 1.71 \end{bmatrix}$$

Average scalar product matrix $\bar{B}$

$$\bar{B} = \begin{bmatrix} 5.0 & 8.0 & 2.0 & 2.0 \\ & 13.0 & 3.0 & 4.0 \\ & & 1.0 & 0.0 \\ \text{sym.} & & & 4.0 \end{bmatrix}$$

Gram factor A^*

$$A^* = \begin{bmatrix} 2.20 & -.38 \\ 3.60 & -.17 \\ .81 & -.59 \\ 1.18 & 1.61 \end{bmatrix}$$

TABLE 1 (continued)

$$C_k^* = (A^{*\prime}A^*)^{-1}A^{*\prime}\, B_k A^*(A^{*\prime}A^*)^{-1}$$

$$\begin{array}{lccc} k: & 1 & 2 & 3 \\ & \begin{bmatrix} 1.32 & -.24 \\ -.24 & 1.18 \end{bmatrix} & \begin{bmatrix} .57 & .10 \\ .10 & .63 \end{bmatrix} & \begin{bmatrix} 1.11 & .14 \\ .14 & 1.19 \end{bmatrix} \end{array}$$

Solution

$$V_1 = \begin{bmatrix} .81 & .59 \\ -.59 & .81 \end{bmatrix} \qquad A = A^*V_1 = \begin{bmatrix} 2.0 & 1.0 \\ 3.0 & 2.0 \\ 1.0 & .0 \\ .0 & 2.0 \end{bmatrix}$$

$D_k^2 = V_1\, C_k^*\, V_1$ (subjective metrics in diagonal):

$$\begin{array}{lccc} k: & 1 & 2 & 3 \\ & D_1^2 = \begin{bmatrix} 1.5 & .0 \\ .0 & 1.0 \end{bmatrix} & D_2^2 = \begin{bmatrix} .5 & .0 \\ .0 & .7 \end{bmatrix} & D_3^2 = \begin{bmatrix} 1.0 & .0 \\ .0 & 1.3 \end{bmatrix} \end{array}$$

rather orthogonal matrices, they still would return the correct order of the roots in (4.8) because the orthogonal portions of these matrices cancel, so to speak. However, if all C_k have multiple roots, then it is conceivable that the stronger model may become insoluble even in the exact case. But such a possibility is probably too remote to invest much study in.

Let us now turn to zero roots. They are usually not ruled out per definition by the proponents of the stronger model. They argue [*e.g.*, Carroll & Chang, 1970, p. 285] that such zero roots have a clear-cut psychological interpretation. Psychologically zero roots would mean that the corresponding dimensions in A are not utilized by that given subject. Technically they would mean that only part of A is recoverable from the responses of that given subject, which is not serious because the rest of A may be recoverable from the responses of other subjects. However, one may also ask what they mean geometrically. Geometrically they would mean that the metrics in (1.2), (1.3) become improper with the effect that the first distance axiom is weakened. Distinct points may now have zero "distance." In terms of the norms (2.2) and the associated scalar products this means that non-null vectors can have zero length and, thus, be orthogonal to themselves. Some may argue that this loss in our geometric intuitions is too high a price to pay for zero roots, and they may prefer to rule them out by definition. In this case the model would have to be rejected should they ever arise in practice.

In applications to fallible data we would expect the solution for A in (3.4) to be fairly robust because it is based on a group average. We would recommend to continue the analysis with $\hat{A}_k^*$ in (3.8), eliminating, if necessary, subjects for which (i) was not met, so as to ensure that it is met exactly for the remaining subjects. The main technical complication in the fallible case is the problem of selecting B_1 in (4.4). Since we can pivot the analysis

TABLE 2

Numerical Illustration: Fallible Case

Coordinate matrices $A_k = AD_k^2 + E_k^*$, where A and D_k^2 are as in Table 1 and E_k^* is error:

$$\underset{A_1}{\begin{bmatrix} 1.05 & 2.40 \\ 1.95 & 3.60 \\ .05 & 1.25 \\ 2.05 & .02 \end{bmatrix}} \quad \underset{A_2}{\begin{bmatrix} .80 & 1.40 \\ 1.70 & 2.00 \\ -.03 & .70 \\ 1.70 & .05 \end{bmatrix}} \quad \underset{A_3}{\begin{bmatrix} 1.00 & 2.10 \\ 2.10 & 2.90 \\ .10 & 1.10 \\ 2.40 & .10 \end{bmatrix}}$$

Scalar product matrices B_k Gram factors A_k^*

$$B_1 = \begin{bmatrix} 6.86 & 10.69 & 3.05 & 2.20 \\ & 16.76 & 4.60 & 4.07 \\ & & 1.57 & .13 \\ \text{sym.} & & & 4.20 \end{bmatrix} \qquad A_1^* = \begin{bmatrix} 2.60 & -.29 \\ 4.09 & -.11 \\ 1.11 & -.58 \\ 1.04 & 1.77 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 2.60 & 4.16 & .96 & 1.43 \\ & 6.89 & 1.35 & 2.99 \\ & & .49 & -.02 \\ \text{sym.} & & & 2.89 \end{bmatrix} \qquad A_2^* = \begin{bmatrix} 1.56 & -.42 \\ 2.62 & -.21 \\ .47 & -.52 \\ 1.24 & 1.17 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 5.41 & 8.19 & 2.41 & 2.61 \\ & 12.82 & 3.40 & 5.33 \\ & & 1.22 & .35 \\ \text{sym.} & & & 5.77 \end{bmatrix} \qquad A_3^* = \begin{bmatrix} 2.25 & -.59 \\ 3.57 & -.27 \\ .90 & -.63 \\ 1.63 & 1.77 \end{bmatrix}$$

Average scalar product matrix $\bar{B}$ Gram factor A^*

$$\bar{B} = \begin{bmatrix} 4.96 & 7.68 & 2.14 & 2.08 \\ & 12.16 & 3.12 & 4.13 \\ & & 1.09 & .15 \\ \text{sym.} & & & 4.29 \end{bmatrix} \qquad A^* = \begin{bmatrix} 2.18 & -.43 \\ 3.48 & -.18 \\ .87 & -.58 \\ 1.27 & 1.63 \end{bmatrix}$$

TABLE 2 (continued)

Test of first constraint: space P_A

$$P_A = \begin{bmatrix} .30 & .42 & .18 & -.07 \\ & .64 & .19 & .14 \\ & & .14 & -.24 \\ \text{sym.} & & & .91 \end{bmatrix}$$

Residuals $E_k = (I - P_A)A_k^*$ L. S. fit $\hat{A}_k^* = P_A A_k^*$ $k = 1,N$

$$\underset{E_1}{\begin{bmatrix} -.02 & .08 \\ .03 & -.05 \\ -.04 & -.01 \\ -.02 & .01 \end{bmatrix}} \quad \underset{E_2}{\begin{bmatrix} -.01 & -.03 \\ .03 & .04 \\ -.07 & -.05 \\ -.02 & -.02 \end{bmatrix}} \quad \underset{E_3}{\begin{bmatrix} .03 & -.06 \\ -.05 & .03 \\ .09 & .03 \\ .03 & .00 \end{bmatrix}} \quad \underset{\hat{A}_1^*}{\begin{bmatrix} 2.63 & -.36 \\ 4.06 & -.06 \\ 1.14 & -.57 \\ 1.06 & 1.75 \end{bmatrix}} \quad \underset{\hat{A}_2^*}{\begin{bmatrix} 1.56 & -.39 \\ 2.59 & -.25 \\ .54 & -.46 \\ 1.26 & 1.19 \end{bmatrix}} \quad \underset{\hat{A}_3^*}{\begin{bmatrix} 2.22 & -.53 \\ 3.62 & -.30 \\ .81 & -.67 \\ 1.59 & 1.77 \end{bmatrix}}$$

$$C_k^* = (A^{*\prime}A^*)^{-1}A^{*\prime}\, B_k A^*(A^{*\prime}A^*)^{-1}$$

k: 1, 2, 3

$$\begin{bmatrix} 1.33 & -.25 \\ -.25 & 1.15 \end{bmatrix} \quad \begin{bmatrix} .57 & .12 \\ .12 & .60 \end{bmatrix} \quad \begin{bmatrix} 1.10 & .14 \\ .14 & 1.24 \end{bmatrix}$$

Solution:

$$V_1 = \begin{bmatrix} .82 & .58 \\ -.58 & .82 \end{bmatrix} \qquad A = A^*V_1 = \begin{bmatrix} 2.03 & .91 \\ 2.95 & 1.86 \\ 1.04 & .02 \\ .10 & 2.07 \end{bmatrix}$$

Test of second constraint - (subjective metrics in diagonal):

$$D_k^2 = V_1' C_k^* V_1$$

k: 1, 2, 3

$$\begin{bmatrix} 1.51 & .00 \\ .00 & .98 \end{bmatrix} \quad \begin{bmatrix} .47 & .02 \\ .02 & .70 \end{bmatrix} \quad \begin{bmatrix} 1.02 & -.02 \\ -.02 & 1.32 \end{bmatrix}$$

on any one of the given N subjects, we might find that the matrices $V_1'C_k^*V_1$ are not close to being diagonal if subject A is chosen as subject 1, with a different finding for subject B. This injects an element of subjectivity into the practical task of testing the constraint (ii) of the stronger model. Short of carrying out the analysis for all N subjects, pivoting on each one in turn, no wholly satisfactory solution of this problem has come to our attention, although several of our reviewers have volunteered suggestions. Ideally one might wish to base this decision, just as checking the first constraint, on some sort of group average. How exactly this can be accomplished, if at all, is yet an open question.

Another practical problem is how to decide with real data at which point, exactly, a diagonal matrix ceases to be diagonal. This decision must be faced and resolved in some manner, however subjectively, in any case, as long as one insists on a testable model. The present solution simply renders the built in constraints of the Bloxom, Horan, Carroll and Chang model explicit.

Finally, a word about the more general model. Some may be tempted to think that this model should be preferred simply because it is more general, and, perhaps also, because its solution is so simple. But these are relatively minor points compared with the overriding question of what it promises as a research tool. An answer to this question depends, in part at least, on the intuitive appeal of the model, which not only bears on the chances of it ever being verified in the real world (which, of course, are better for the more general model) but also, and more importantly, on the expected benefits if it ever does fit. The special case considered by Bloxom, Horan, Carroll and Chang seems far superior in this respect.* Our purpose in considering the more general formulation was mainly to be able to use its solution as a tool for a simple algebraic solution of the model by Bloxom, Horan, Carroll and Chang.

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* We are advised by one of our reviewers that Tucker (in press) presents a case for the more general model in a forthcoming paper, which we have not yet had the opportunity to see.