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A UNIQUE VARIANCE MODEL FOR THREE-MODE FACTOR ANALYSIS

BY

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THESIS

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Psychology in the Graduate College of the University of Illinois, 1969

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY

SUPERVISION BY FRANK WESLEY SNYDER

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BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF DOCTOR OF PHILOSOPHY

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TABLE OF CONTENTS

LIST OF TABLES

THE MODEL

In psychological experiments, data are frequently collected which can be cross-classified by three dimensions or modes, i.e., the data from a rectangular prism. One example could be rating a number of different objects using the same rating scales with different methods (Kelley and Fiske, 1951). Another example would be having a number of different raters appraise concepts on Semantic Differential scales (Osgood, et al., 1957). Other examples include a complex tracking task where several measures of performance at several stages of practice are obtained on the subjects (Parker and Fleishman, 1960) and clinical judgments of traits associated with schizophrenia in which experienced judges rated the schizophrenicity of WAIS vocabulary and comprehension items (Mills and Tucker, 1965). Until quite recently there has been no data analysis procedure aimed at uncovering the basic dimensions of the data that permitted the investigator to do more than analyze the various two-way classifications which could be derived from the observed three-mode data matrix. The derived matrices have been formed by an averaging or summing technique, or by dividing the prism into planes and analyzing the resultant two-mode matrices. All of these derived forms of analysis have, inherent in them, problems of interpretation and organization (Levin, 1965; Hoffman and Tucker, 1964; Snyder and Wiggins, 1968).

In order to handle data classified in three modes, Tucker (1963a, 1963b, 1966), developed the three-mode factor analysis model. The basic model can be expressed as

(1) $$x_{ijk} = \sum_{m}\sum_{p}\sum_{q} f_{im} b_{jp} c_{kq} g_{mpq} + e_{ijk}$$

where f_{im} is the i^{th} individual's loading on the m^{th} individual factor, b_{jp} is j^{th} trait's loading on the p^{th} trait factor, C_{kq} is the k^{th} method's loading on the q^{th} method factor, g_{mpq} is the core coefficient which interrelates the m^{th} individual factor, the p^{th} trait factor, and the q^{th} method factor, and e_{ijk} is the error of fit to the observed data point x_{ijk}. Rewriting Equation 1 using the Kronecker product and combination variable notations (Tucker, 1966; a summary of these notational conveniences appears in the Appendix), we have the matrix equation

(2) $${}_{i}X_{(jk)} = {}_{i}F_{m}G_{(pq)}({}_{p}B_{j}\overline{\underline{X}}_{q}C_{k}) + {}_{i}E_{(jk)}$$

where X, F, B, C, G, and E are matrices of order I x J(K), I x M, J x P, K x Q, M x P(Q), and I x J(K) respectively whose elements are the x_{ijk}'s, f_{im}'s, b_{jp}'s, c_{kq}'s, g_{mpq}'s, and e_{ijk}'s discussed above. For convenience in the discussion that will follow an X^{c} matrix is defined, such that

(3) $${}_{i}X^{c}_{(jk)} = {}_{i}F_{m}G_{(pq)}({}_{p}B_{j}\overline{\underline{X}}_{q}C_{k}) .$$

It is also convenient to define a matrix R, such that

(4) $${}_{(jk)}R_{(jk)} = {}_{(jk)}X_{i}X_{(jk)}$$

and a matrix R^{c}, such that

(5) $$_{(jk)}R^{c}_{(jk)} = {}_{(jk)}X^{c}_{i}\,X^{c}_{(jk)}$$

where R and R^{c} will be sums-of-squares and cross-products or variance-covariance or correlation matrices depending on the particular scaling of the X and X^{c} matrices. By combining Equations 2, 3, 4, and 5, the model can be expressed in its correlational form as

(6) $$_{(jk)}R_{(jk)} = ({}_{j}B_{p}\,\overline{X}_{k}\,C_{q})_{(pq)}G_{m}\,F_{i}\,F_{m}\,G_{(pq)}\,({}_{p}B_{j}\,\overline{X}_{q}\,C_{k}) + {}_{(jk)}\delta_{(jk)}$$

or

(7) $$_{(jk)}R_{(jk)} = {}_{(jk)}R^{c}_{(jk)} + {}_{(jk)}\delta_{(jk)}$$

where

(8) $$_{(jk)}\delta_{(jk)} = {}_{(jk)}E_{i}\,E_{(jk)} .$$

Tucker's model, as represented in Equations 1-8, contains some rather severe restrictions. In its present form the model is analogous to a principal components resolution of a two-mode matrix. Using the model it is possible to estimate the m largest individual factors, the p largest trait factors, the q largest method factors, and the core matrix G, but no provision has been made for isolating or estimating the effects of any of the trait or method specific factors, which may have influenced the measurements obtained. In other words, the model makes no provision for communality estimation nor an investigation of the common factor space.

In more recent work Hoffman and Tucker (1964), and Tucker (1966), have developed a model that allows for the estimation of specific factors associated with the combination variables. In matrix form this model can be expressed as

$$(9) \qquad {}_{i}X_{(jk)} = {}_{i}X^{c}_{(jk)} + {}_{i}X^{\gamma}_{(jk)} + {}_{i}E_{(jk)}$$

where X^{c} refers to that part of the observed scores due to the factors remaining after the combination variable specifics have been removed, X^{γ} refers to that part of the observed scores due to the combination variable specifics (a combination of a specific trait k and method j), and E refers to the error of fit to the data. In order to make the model workable it is necessary to define certain properties of the X^{c}, X^{γ}, and E matrices. It is desirable for the common parts scores to be uncorrelated with the unique parts scores, that is

$$(10) \qquad {}_{(jk)}X^{c}_{i}X^{\gamma}_{(jk)} = 0 \; ;$$

and for the errors of fit to be uncorrelated with both the unique and common parts scores

$$(11) \qquad {}_{(jk)}X^{l}_{i}E_{(jk)} = 0; \quad l = c, \gamma \, .$$

Using Equations 9, 10, and 11 the correlation matrix R can be expressed as the sum of three terms

$$(12) \qquad {}_{(jk)}R_{(jk)} = {}_{(jk)}X^{c}_{i}X^{c}_{(jk)} + {}_{(jk)}X^{\gamma}_{i}X^{\gamma}_{(jk)} + {}_{(jk)}E_{i}E_{(jk)}$$

or

$$(13) \qquad {}_{(jk)}R_{(jk)} = {}_{(jk)}R^{c}_{(jk)} + {}_{(jk)}\Gamma_{(jk)} + {}_{(jk)}\delta_{(jk)}$$

where ${}_{(jk)}\Gamma_{(jk)}$ is a diagonal matrix

$$(14) \qquad {}_{(jk)}\Gamma_{(jk)} = {}_{(jk)}X_i^{\gamma}X^{\gamma}_{(jk)}$$

and R^c and δ are defined as in Equations 5 and 8 respectively.

This model represents an improvement over the model mentioned previously in that it is possible to estimate Γ and hence remove the influence of the combination variable specifics from our observed correlation matrix. However, the problem still remains of estimating and removing the influences of the trait specific and the method specific factors.

Hoffman and Tucker (1964) applied the above model to a multitrait-multimethod (Campbell and Fiske, 1959) matrix collected by Kelley and Fiske (1951). Their findings largely substantiate Kelley and Fiske's findings, of four well defined trait factors, in addition to providing information concerning the data modes not previously investigated. The results produced with the model in the analysis mentioned above and other results produced with the first model (Tucker, 1967; Mills and Tucker, 1964) in various contexts have been gratifyingly reasonable in the light of information obtained using different analytic procedures. However, there have been problems raised recently by Horn and Cattell (1965) and by Campbell and O'Connell (1967) which these models have been unable to handle. The shortcomings of the models rest mainly on the fact that in one it is not possible to estimate any specific influences and in the other it is possible only to estimate the specifics associated with the combination variables. Horn and Cattell, working in the general area of measurement of motivation, noted that there is a problem isolating response set specifics and method specifics. Campbell and O'Connell noted that the intertrait correlations are higher when both traits are measured with the same method than when measured with two different methods. A direct outgrowth of these findings is that efforts have been made to estimate the influence of methods factors for each method. The results, thus far, as Campbell and O'Connell report, have been rather disappointing. The conclusion at which the authors eventually arrived is that the observed results can be accounted for only by postulating the existence of method specific factors. Another empirical conclusion, the implications of which will be demonstrated later, is that the contribution of the method specific factors is not constant for all intertrait correlations employing a common method.

It is the purpose of the present investigation to develop a data analysis model, for three-mode matrices, which will allow the isolation and estimation of trait specific factors, method specific factors, and combination variable specific factors. Such a model will enable the investigation of the common factor space of the trait and method modes without the contaminating influences of specific factors.

A particular score x_{ijk} is assumed to be the sum of influences of the common factors of the elemental modes i, j, and k, the specific associated with method j, the specific associated with trait k, the specific associated with the combination of method j and trait k, and a random error of fit to the data. In explicating the proposed model, "trait" and "method" are used in reference to the two variable modes which are not the subject mode; however, the model is quite general and need not be restricted to the multitrait-multimethod situation.

The above conditions can be represented in matrix form as

$$(15)\qquad {}_iX_{(jk)} = {}_iX^c_{(jk)} + {}_iX^\alpha_{(jk)} + {}_iX^\psi_{(jk)} + {}_iX^\gamma_{(jk)} + {}_iE_{(jk)}$$

where X^c represents the common factors' influence, X^α the trait specifics' influences, X^ψ the method specifics' influences, X^γ the combination variable specifics' influences, and E the error of fit to the data. It will be assumed that the common parts' scores are uncorrelated with the specific parts' scores and with the errors of fit. Further, it will be assumed that all specific parts' influences are mutually uncorrelated and are uncorrelated with the errors of fit. The above conditions can be summarized in matrix form as

$$(16)\qquad {}_{(jk)}X^l{}_iX^{l'}_{(jk)} = 0; \quad l = c, \alpha, \psi, \gamma; \quad l' = c, \alpha, \psi, \gamma; \quad l \neq l'$$

$$(17)\qquad {}_{(jk)}X^l{}_lE_{(jk)} = 0; \quad l = c, \alpha, \psi, \gamma .$$

Employing the restrictions set forth in Equations 16 and 17 and scaling appropriately, Equation 15 implies that

$$(18)\quad {}_{(jk)}R_{(jk)} = {}_{(jk)}R^{c}_{(jk)} + {}_{(jk)}\nabla_{(jk)} + {}_{(jk)}\psi_{(jk)} + {}_{(jk)}\Gamma_{(jk)} + {}_{(jk)}\delta_{(jk)}$$

where

$$(19)\quad {}_{(jk)}R^{c}_{(jk)} = {}_{(jk)}X_i^{c}X^{c}_{(jk)}$$

$$(20)\quad {}_{(jk)}\nabla_{(jk)} = {}_{(jk)}X_i^{\alpha}X^{\alpha}_{(jk)}$$

$$(21)\quad {}_{(jk)}\psi_{(jk)} = {}_{(jk)}X_i^{\psi}X^{\psi}_{(jk)}$$

$$(22)\quad {}_{(jk)}\Gamma_{(jk)} = {}_{(jk)}X_i^{\gamma}X^{\gamma}_{(jk)}$$

$$(23)\quad {}_{(jk)}\delta_{(jk)} = {}_{(jk)}E_iE_{(jk)} .$$

Now, consider one of the submatrices of R, a ${}_{j}R_{j'}$ matrix. This matrix is composed of the intertrait correlations obtained using two methods j and j'. Analogous to the classical factor analytic model, it is not unreasonable to assume that each trait k has some specific component which it shares with no other measure k' $(k \neq k')$. In terms of the correlation matrix and the factor analytic model this would mean that ${}_{j}R_{j'}$ can be represented as

$$(24)\quad {}_{j}R_{j'} = {}_{j}R^{c}_{j'} + t_{jj'}(\Phi) + {}_{j}\delta_{j'}$$

where Φ is a diagonal matrix with entries $\phi_1, \phi_2, \ldots, \phi_k$ and where $t_{jj'}$ is constant for the ${}_{j}R_{j'}$ subsection. Thus far only one of the J^2 possible submatrices of R have been considered. It is possible to advance the argument that any ${}_jR_{j'}$ submatrix is composed as in Equation 12, but this would seem to be an unnecessarily restrictive assumption and it seems more fruitful to consider possible ways of representing the influences of the selection of the particular methods j and j'. To this end, assume that the submatrices ${}_jR_{j'}$ are composed in the following fashion:

(25) $${}_jR_{j'} = {}_jR^c_{j'} + a_j a_{j'} (\Phi) + {}_j\delta_{j'}$$

where

(26) $$a_j a_{j'} = t_{jj'}$$

or

(27) $${}_jR_{j'} = {}_jR^c_{j'} + {}_j\nabla_{j'} + {}_j\delta_{j'}$$

where Φ is the diagonal matrix mentioned above and the a_j's are scalers. At this point it is worthwhile to note, that by identifying the $a_j a_{j'}(\Phi)$ matrix with the method specific influences and by allowing the product $a_j a_{j'}$ to vary with varying j and j', Campbell and O'Connell's necessary requirement of a model to adequately explain multitrait-multimethod matrices has been extended to include trait specifics of the same general nature. A matrix ∇ is now defined as being composed of submatrices ${}_j\nabla_{j'}$ each of which is a diagonal matrix $a_j a_{j'}(\Phi)$. Rearranging the element of ∇ so that the indices are nested j within k rather than k within j, as they were originally, the possibility of a further generalization of the model becomes evident.

Up to this point the ∇ matrix has been discussed under the implicit assumption that each ${}_{k}\nabla_{k'}$ submatrix (j within k) was of unit rank. In order to allow for differential overlappings of trait specific effects given a particular pair of methods j and j' the assumption of unit rank will be discarded in favor of a more general specification of the elements of the matrix. This more general specification can be represented as

$$(28) \qquad {}_{(jk)}R_{(jk)} = {}_{(jk)}R^{c}_{(jk)} + {}_{(jk)}\nabla_{(jk)} + {}_{(jk)}\delta_{(jk)}$$

where

$$(29) \qquad \alpha_{jj'k} = \sum_{r} \phi_{kr} a_{jr} a_{j'r} \, .$$

Specifying the ∇ matrix as in Equations 28 and 29 allows for the possibility (but does not require) that trait specific effects are influenced by more than one specific factor.

In considering the composition of the Ψ matrix the same logic is used as with the ∇ matrix. The ψ's should be allowed to vary with varying k and k'. Also the ${}_{j}\Psi_{j'}$ (k within j) submatrices should be defined generally enough as to evidence rank greater than one if that is the true state of affairs. To this end the ψ's are defined as

$$(30) \qquad \psi_{kk'j} = \sum_{r'} \lambda_{jr'} d_{kr'} d_{k'r'}$$

where the product $d_{kr'} d_{k'r'}$ varies with varying k and k' and where

$\lambda_{jr'}$ is a constant for each combination of j and r'. It is now possible to add another term to Equation 28, that is, R can be defined as

$$(31) \quad {}_{(jk)}R_{(jk)} = {}_{(jk)}R^{c}_{(jk)} + {}_{(jk)}\nabla_{(jk)} + {}_{(jk)}\Psi_{(jk)} + {}_{(jk)}\delta_{(jk)} .$$

Specifically the function of the Ψ matrix is to define method specific influences and to allow these influences to vary for varying intertrait correlations.

Only one last matrix, Γ , remains to be specified. Thus far, the R matrix has been expressed as a sum of the common parts correlations, the trait specific influences, and the method specific influences. It is quite possible, however, that the specific influences present in the main diagonals of R (both a common trait and a common method) are not merely a sum of the specific influences of traits and methods but have in addition another influence which occurs only when we have measured the same trait using the same method. Thus, it seems desirable to postulate the existence, in the model, of a diagonal matrix Γ of these combination variable influences. The point has been reached when it is now possible to specify the correlation matrix R as was done in Equation 18, repeated here for the reader's convenience:

$$(32) \quad {}_{(jk)}R_{(jk)} = {}_{(jk)}R^{c}_{(jk)} + {}_{(jk)}\nabla_{(jk)} + {}_{(jk)}\Psi_{(jk)} + {}_{(jk)}\Gamma_{(jk)} + {}_{(jk)}\delta_{(jk)}$$

where δ is the matrix containing the error of fit to the observed correlation matrix.