Maximal Rank of an Element of a Tensor Product

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ABSTRACT

Upper bounds are given for the maximal rank of an element of the tensor product of three vector spaces.

1. BACKGROUND

Let $X_1, X_2, \ldots, X_n$ be vector spaces of finite dimension $d_1, d_2, \ldots, d_n$ over some field K. Let X be the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$. Then a vector x in X is defined to be *decomposable* [8] if there exist vectors x_i in X_i (for $i = 1, 2, \ldots, n$) such that $x = x_1 \otimes x_2 \otimes \cdots \otimes x_n$. The *rank* of a general vector x is defined to be the smallest integer $r_K(x)$ such that x is a sum of $r_K(x)$ decomposable vectors: see [4, II, §7, No. 8].

We write r_K rather than r to emphasize the dependence on the field K. If L is a field containing K, and Y_i is the L-vector space $X_i \otimes_K L$ for $i = 1, 2, \ldots, n$, then we may reasonably regard an element x of X as an element of $Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n$. If $n = 2$ then $r_K(x) = r_L(x)$ (see [4, II, §7,

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No. 9]), but [2, 13] give examples with $n = 3$, $K = \mathbf{R}$, and $L = \mathbf{C}$ in which $r_L(x) < r_K(x)$.

The *maximal rank* of X is defined to be

$$\max_{x \in X} \{r_K(x)\}.$$

Since this depends on X only through the dimensions $d_1, d_2, \ldots, d_n$ and the field K, we define the maximal-rank function f_n by

$$f_n(d_1, d_2, \ldots, d_n; K) = \max\{(r_K(x)) : x \in K^{d_1} \otimes K^{d_2} \otimes \cdots \otimes K^{d_n}\}.$$

It seems plausible that, in general, the values of f_n may also change with K, although we have no evidence of this yet. However, it is well known that, for all fields K,

$$f_2(d_1, d_2; K) = \min\{d_1, d_2\} \tag{1}$$

and

$$f_n(d_1, d_2, \ldots, d_{n-1}, 1; K) = f_{n-1}(d_1, d_2, \ldots, d_{n-1}; K). \tag{2}$$

Various weak bounds are known on the values of f_n (see [9, 13, 15]), such as

$$\frac{d_1 d_2 \cdots d_n}{d_1 + d_2 + \cdots + d_n - n + 1} \leqslant f_n(d_1, d_2, \ldots, d_n; K) \leqslant \frac{d_1 d_2 \cdots d_n}{\max_i d_i}, \tag{3}$$

and

$$\max\left\{\min\left\{\prod_{i \in I} d_i, \prod_{i \notin I} d_i\right\} : \varnothing \subset I \subset \{1, 2, \ldots, n\}\right\} \leqslant f_n(d_1, d_2, \ldots, d_n; K). \tag{4}$$

However, for $n \geqslant 3$, very little is known about the exact values of f_n for general fields except for the result of Ja'Ja' in [10]:

$$f_3(2, d, e; K) = \min\{d, e\} + \min\{\min\{d, e\}, \lfloor \tfrac{1}{2} \max\{d, e\} \rfloor\} \tag{5}$$

for all sufficiently large fields K. "Sufficiently large" is defined by a slowly growing function of d and e: in particular, Equation (5) is true whenever K is infinite. It has the important special case

$$f_3(2, 2, 2; K) = 3. \tag{6}$$

Further, Ja'Ja' shows that

$$f_3(d, d, e; K) \leqslant \frac{3de}{4} \tag{7}$$

for all sufficiently large fields K.

Atkinson and his coworkers [1, 3] give some upper bounds on f_3 for algebraically closed fields K:

$$f_3(d, d, e; K) \leqslant \binom{e+1}{2} d, \tag{8}$$

$$f_3(d_1, d_2, e; K) \leqslant \min\{d_1, d_2\} + \left\lfloor \frac{e}{2} \right\rfloor \max\{d_1, d_2\}, \tag{9}$$

$$f_3(d_1, d_2, d_1 d_2 - u) = d_1 d_2 - \left\lceil \frac{u}{2} \right\rceil \quad \text{if} \quad u \leqslant \min\{4, d_1, d_2\}. \tag{10}$$

One application of such bounds is to the approximation of arrays of data by low-rank tensors. Even though the *typical rank* introduced in [14] may be more relevant to such questions, there is still some statistical interest in determining the values of f_n for $K = \mathbf{R}$, especially when $n = 3$: see [5–7, 11, 13]. It is not known whether (8)–(10) hold when $K = \mathbf{R}$. Thus we feel it worthwhile to present an upper bound on f_3 which is an improvement on (3) but which has a fairly short proof. Henceforth we abbreviate f_3 to f.

2. RESULTS

THEOREM 1. *Let K be any field. If $d_1 \geqslant 2$ and $d_2 \geqslant 2$ then*

$$f(d_1, d_2, d_3; K) \leqslant d_3 + f(d_1 - 1, d_2 - 1, d_3; K).$$

Proof. Let $\{x_1, x_2, \ldots, x_{d_1}\}$, $\{y_1, y_2, \ldots, y_{d_2}\}$, and $\{z_1, z_2, \ldots, z_{d_3}\}$ be bases for X_1, X_2, and X_3 respectively. Then every vector x in X has the form

$$\sum_{i,j,k} \alpha_{ijk} x_i \otimes y_j \otimes z_k$$

for some α_{ijk} in K. The idea of the proof is as follows. If any α_{ijk} is nonzero, then we can find a decomposable vector which agrees with x on two directions through the cell (i, j, k) and takes any prescribed values on the third direction. Making these prescribed values nonzero enables us to repeat the process at the other cells in the third direction thrugh (i, j, k) until both of the 2-dimensional slabs through this direction have been reduced to zero.

Since the zero vector is itself decomposable, we may assume that x is nonzero and hence, without loss of generality, that $\alpha_{111} \neq 0$. Put

$$u_1 = \sum_i \alpha_{i11} x_i$$

$$v_1 = \sum_j \alpha_{1j1} y_j$$

$$w_1 = \alpha_{111}^{-1} z_1 + \sum_{k \geqslant 2} \alpha_{111}^{-2} (\alpha_{11k} - 1) z_k,$$

and let $x' = x - u_1 \otimes v_1 \otimes w_1$. Thus if $x' = \sum_{i,j,k} \beta_{ijk} x_i \otimes y_j \otimes z_k$ then

$$\beta_{i11} = 0 \qquad \text{for all } i,$$

$$\beta_{1j1} = 0 \qquad \text{for all } j,$$

$$\beta_{11k} = 1 \qquad \text{for} \quad k \geqslant 2.$$

For $k = 2, 3, \ldots, d_3$, put

$$u_k = \sum_i \beta_{i1k} x_i,$$

$$v_k = \sum_j \beta_{1jk} y_j,$$

and let $x'' = x' - \sum_{k=2}^{d_3} u_k \otimes v_k \otimes z_k$. Then if $x'' = \sum_{i,j,k} \gamma_{ijk} x_i \otimes y_j \otimes z_k$ we have

$$\gamma_{i1k} = 0 \qquad \text{for all } i \text{ and } k,$$

$$\gamma_{1jk} = 0 \qquad \text{for all } j \text{ and } k.$$

In other words, $x'' \in X_1' \otimes X_2' \otimes X_3$, where X_1' is spanned by $\{x_2, \ldots, x_{d_1}\}$ and X_2' is spanned by $\{y_2, \ldots, y_{d_1}\}$. Thus x'' is a sum of at most $f(d_1 - 1, d_2 - 1, d_3; K)$ decomposable vectors, and we have constructed x'' so that $x - x''$ is a sum of (at most) d_3 decomposable vectors. ■

The following corollary gives an explicit bound on f in certain special cases. It is proved by repeated application of Theorem 1 until (1) and (2) can be used.

COROLLARY 2. *If $d_3 = d_1 + d_2 - 2u$ and $0 \leqslant u < \min\{d_1, d_2\}$ then*

$$f(d_1, d_2, d_3) \leqslant d_1 d_2 - u^2,$$

while if $d_3 = d_1 + d_2 - (2t + 1)$ and $0 \leqslant t < \min\{d_1, d_2\}$ then

$$f(d_1, d_2, d_3) \leqslant d_1 d_2 - t(t + 1).$$

Proof. Applying Theorem 1 u times and then using (3) gives

$$\begin{aligned} f(d_1, d_2, d_3; K) &\leqslant ud_3 + f(d_1 - u, d_2 - u, d_3; K) \\ &\leqslant ud_3 + (d_1 - u)(d_2 - u) \\ &= u(d_1 + d_2 - 2u) + (d_1 - u)(d_2 - u) \\ &= d_1 d_2 - u^2. \end{aligned}$$

The second case is proved similarly. ■

Corollary 2 can be rephrased in the following symmetric form.

COROLLARY 3. *If $d_1 \leqslant d_2 \leqslant d_3$ and $d_3 \leqslant d_1 + d_2$ then*

$$f(d_1, d_2, d_3; K) \leqslant \left\lfloor \frac{(d_1 + d_2 + d_3)^2 - 2(d_1^2 + d_2^2 + d_3^2)}{4} \right\rfloor.$$

3. DISCUSSION

Since it clear that $f(2, 2, 2; K) > 2$, our results give a rather quick proof of (6). However, our bounds, although much better than (3) in general, are far from being sharp. For example, Kruskal (personal communication) has shown that $f(3, 3, 3; \mathbf{R}) = 5$, while Corollary 3 gives only $f(3, 3, 3; K) \leqslant 7$. Atkinson and Stephens [3] claim that $f(3, 3, 3; K) = 5$ for all algebraically closed fields K, but details of their proof are omitted.

For algebraically closed fields, Corollary 3 is no improvement on (8) and (9) unless the upper bound in (3) makes both of the other bounds superfluous. For example, if $d \leqslant e \leqslant 2d$ then the upper bounds b, b_{J}, b_{AS}, and b_{BR} on $f(d, d, e; K)$ given by (3), (7), (8), and Corollary 3 are

$$b = d^2, \qquad b_{\mathrm{J}} = \frac{3de}{4}, \qquad b_{\mathrm{AS}} = d\left(\frac{e+1}{2}\right), \qquad b_{\mathrm{BR}} = \left\lceil \frac{e(4d-e)}{4} \right\rceil.$$

Then $b_{\mathrm{BR}} \leqslant b_{\mathrm{J}}$ except in the case that $e = d$ and d is odd, so Corollary 3 is an improvement over (7) for all fields. However, $b_{\mathrm{AS}} \leqslant b_{\mathrm{BR}}$ unless $e = 2d$, in which case $b = b_{\mathrm{BR}} < b_{\mathrm{AS}}$. Thus Corollary 3 would not be useful if the results (8) and (9) could be extended to fields which are not algebraically closed.

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