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Generalized multimode latent variable models: Implementation by standard programs

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Abstract: Three-mode models in factor analysis have not been used very frequently due in part to their mathematical, statistical, and computational complexity. It is shown that standardly-available computer programs such as LISREL and EQS can be used to estimate and test such models. The models are generalized to permit more complex measurement structures, as well as to allow linear structural regressions among the latent variables. These generalized multimode models can be similarly easily computationally implemented. An example is used to illustrate the ideas.

Keywords: Multimode factor analysis, Latent variables, Structural models, Kronecker product models.

1. Introduction

When the design of a measurement instrument, such as psychological ratings, permit a systematic crossing of various facets of content to produce a set of variables, the resulting variables are said to be of the multimode type. An example would be the measurement of four personality traits such as extraversion, anxiety, impulsivity, and motivation by each of three methods, such as peer, teacher, and self-ratings. This design would produce $4 \times 3 = 12$ variables. A random sample assessed on these 12 variables would produce, in Tucker's [12] terminology, three mode data, where the modes are traits, methods, and subjects. Tucker treated these modes interchangeably, leading to exploratory procedures

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for data analysis that are least-squares algebraic and relatively straightforward to implement [8]. These procedures, however, have no statistical basis. Bloxom [5] noted that the sampling mode can be considered random, and that statistical analysis can focus on the two fixed design modes yielding a modified factor analytic model. Extending the model to include both exploratory and confirmatory variants, Bentler and Lee [3] developed statistical methods for three mode data that permit testing the adequacy of any specific parameterization. However, practical analysis of such data has been hindered by the extreme complexity of the methodology. For example, Bentler and Lee presented $1\frac{1}{2}$ pages of matrix equations required for implementing a Gauss–Newton procedure to estimate the parameters of their model, and they made no program publically available to do the estimation. While Tracy and Jinadasa [11] provided a simplified derivation for the Bentler–Lee equations, the resulting derivatives still require three or four-level Kronecker products and, thus, are not usable in practice because they require very extensive computer storage. It is no wonder that these statistical models have not been used very much.

This paper considers several new generalized latent variable models for multimode data and applies the BMDP package program EQS [2] to estimating the parameters of the models and evaluating the goodness of fit of the models to data. Previous approaches to multimode data have been nonstatistical in nature, limited in the number of modes that were considered, have ignored scale-invariant representations, have dealt with a limited set of matrix products, and have not dealt with structural relations among latent or constructed variables. The models considered here allow arbitrary numbers of modes, allow scale-invariant representations when desired, deal with arbitrary matrix products, and allow structural relations among variables. Although the approach to three-mode models developed by Bentler and Lee [3] was quite general in that it developed the statistical, exploratory and confirmatory aspects of the Tucker-Bloxom approach, estimation and testing of even this simple model was conceived in a framework that utilized a specially developed mathematical, statistical, and programming approach. In contrast, in this paper any of the generalized models is viewed as representing a system of linear structural equations, whose parameters under constraints can be estimated by various standard generalized methods such as EQS and LISREL [6]. Estimation and testing is especially easy with the equations-based computer program EQS which is based on the Bentler–Weeks [4] structural relations model. As a consequence, least squares, generalized least squares, and maximum likelihood estimates based on multivariate normal distributional assumptions on the measured variables are available. However, because EQS also provides estimation and testing under elliptical distributions (symmetric generalizations of normal distributions allowing more or less kurtosis for variables), as well as a statistical method for arbitrary distributions (i.e., a distribution-free procedure), this approach for the first time introduces statistical methods for nonnormally distributed multimode data. Related statistical machinery such as Lagrange Multiplier and Wald tests that are available in EQS thus also become automatically accessible to multimode analysis.

In the next section, the Bentler–Weeks model and EQS program are reviewed. In Section 3, some general models for multimode data and their implementation via EQS are discussed. An illustrative example is reported in Section 4, and a concluding discussion is presented in Section 5. A major point of this paper is to show that no new implementation concepts are needed, once the conceptual structure of a model has been developed and its Bentler–Weeks specification generated in an identified form.

2. The Bentler–Weeks model and EQS

A general statistical approach to the analysis of linear structural equation systems based on the Bentler–Weeks [4] model is given as

$$\eta = \beta\eta + \gamma\xi, \tag{1}$$

where β contains the coefficient for regression of "dependent" η variables on each other and γ contains the coefficient for the regression of η variables on ξ variables. Only the "independent" variables ξ have variances and covariances as parameters of the model. Assuming all variables have zero means, with $\Phi = E(\xi\xi')$ the covariance matrix of the independent variables, the parameter matrices of the model are β, γ and Φ. These matrices generate the covariance structure of any observed or measured variables involved in (η, ξ), which are a subset of (η, ξ) since η and ξ may contain latent variables (hypothetical constructs such as factors), errors in variables, disturbances in equations (in the econometric sense), residuals, i.e., any and all variables that may be involved in a linear system. Let $\nu = (\eta', \xi')'$, B be a partitioned matrix containing rows $(\beta, 0)$ and $(0, 0)$, and $\Gamma = (\gamma', \mathrm{I})'$. Then $\nu = \mathrm{B}\nu + \Gamma\xi$ is equivalent to (1). Assuming the inverse exists, it follows that all variables can be expressed as $\nu = (I - B)^{-1}\Gamma\xi$, namely, as a linear combination of independent variables. Now, letting x be the vector of measured variables which must be a subset of all variables ν, they can be expressed as $x = G_x\nu$ where G_x is a known selection matrix of zeros and ones. Thus, $x = G_x(I - B)^{-1}\Gamma\xi$, and the covariance matrix Σ of x is obtained as $\Sigma = E(xx')$, or

$$\Sigma = G_x(I - B)^{-1}\Gamma\Phi\Gamma'(I - B)^{-1\prime}G_x'. \tag{2}$$

Thus if θ is the vector of free or unknown parameters in β, γ, and Φ, $\Sigma = \Sigma(\theta)$. Considering only the nonredundant elements in Σ, the lower triangle, and placing these into the vector σ, we have $\sigma = \sigma(\theta)$. Evidently, this is a covariance structure model. We shall assume that the means $\mu = E(x)$ are not a function of θ, so that estimation and evaluation of (2) can be done on deviation scores and the sample covariance matrix S.

The EQS [2] program estimates the parameters via the generalized least squares (minimum distance) approach by minimizing a function

$$Q(\theta) = (s - \sigma(\theta))'W(s - \sigma(\theta)), \tag{3}$$

where s is the vector of nonredundant sample covariances (selected from S) to be modeled, and $\sigma(\theta)$ is a model for the covariances with the unknown elements taken from the parameter matrices β, γ and Φ. Aside from the fixed parameters, the unknown parameters may be freely estimated, or the free parameters can be estimated within specified intervals, or they may be made to satisfy linear equality constraints. The weight matrix W can be specified in several ways to yield a number of different estimates that depends on the distribution assumed for the variables x (normal, elliptical, arbitrary) and the computational method involved. Under arbitrary distributions, W^{-1} must be a consistent estimator of the asymptotic covariance matrix of the data vector s, so that $Q(\hat{\theta})$ has an asymptotic chi-square distribution to evaluate the structural hypothesis $\Sigma = \Sigma(\theta)$. Under normal distributions on x, W has a simpler form depending only on a consistent estimator of the sample covariance matrix S (to yield best generalized least squares estimates) or $\Sigma(\hat{\theta})$ (iteratively updated, to yield maximum likelihood estimates). The statistical theory is summarized in Chapter 3 of [2]. The output of the program gives the final estimates, standard error estimates, the goodness-of-fit test, and other important information such as Lagrange Multiplier or Wald tests of restrictions, for further statistical inferences. The simple form of eq. (1) makes it clear that any linear structural model can be analyzed by EQS. For example, the factor analytic model $x = A\zeta + \varepsilon$ can be handled by letting $\eta = x$, $\beta = 0$, $\gamma = (A, I)$ and $\xi = (\zeta', \varepsilon')'$. Next, we will demonstrate how EQS can be applied to analyze various multimode models.

3. Multimode latent variable models

Tucker [12] developed a generalization of principal component analysis to more than two modes of measurements. His model was rewritten as a modified factor analytic model by Bloxom [5], and developed statistically by Bentler and Lee [3]. A representation of this model is given by

$$x = \mu + (A \otimes B)G\eta + \zeta, \tag{4}$$

where $\mu = E(x)$ and $\Sigma = E(x - \mu)(x - \mu)'$. Assuming that $E(\eta\zeta') = 0$, the covariance structure is

$$\Sigma = (A \otimes B)G\Phi G'(A \otimes B)' + Z, \tag{5}$$

where $\otimes$ is the Kronecker product of matrices, A and B are factor loading matrices for the two fixed modes, G is a rearranged core matrix, Φ is the covariance matrix of the common factors η, and Z is the arbitrary (not necessarily diagonal) covariance matrix of the unique factors ζ. Many interesting specializations of this model exist, but these will not be discussed here.

Consider the following system of structural equations

$$\begin{aligned} x &= \mu + (A \otimes I)\alpha_1 + \zeta, \\ \alpha_1 &= (I \otimes B)\alpha_2, \\ \alpha_2 &= G\eta, \end{aligned} \tag{6}$$

where α_1 and α_2 are random vectors of appropriate order. Evidently, the equations (6) are equivalent to (4) as can be verified by substitution. To demonstrate that (6) is a special case of (1), let $\eta = ((x-\mu)', \alpha_1', \alpha_2')'$ and $\xi = (\zeta', \eta')'$ be the vectors of (1) in terms of the vectors of (6); let β be the 3×3 supermatrix with rows $(0, A \otimes \mathrm{I}, 0)$, $(0, 0, I \otimes \mathrm{B})$, and $(0, 0, 0)$; and let γ be the 3×2 supermatrix with rows $(I, 0)$, $(0, 0)$, and $(0, G)$. Clearly (6) is a special case of (1), with the covariance structure given in (5) consequently being a special case of (2). As a result, the EQS program can be applied to analyze the model. Note that the Kronecker product matrices in (6) are simply matrices with some known zeros and equality constraints among elements.

Principal component models have been extended to more than three modes [7,9], but this has not been done for factor analytic models. The above approach can be easily extended to an arbitrary number of modes by considering the model

$$x = \mu + (A_1 \otimes \cdots \otimes A_k)G\eta + \zeta \tag{7}$$

with covariance structure

$$\Sigma = (A_1 \otimes \cdots \otimes A_k)G\Phi G'(A_1 \otimes \cdots \otimes A_k)' + Z. \tag{8}$$

The Bentler–Weeks representation of (7) is given by

$$\begin{aligned} x &= \mu + (A_1 \otimes I_2 \otimes \cdots \otimes I_k)\alpha_1 + \zeta, \\ \alpha_1 &= (I_1 \otimes A_2 \otimes \cdots \otimes I_k)\alpha_2, \\ &\vdots \\ \alpha_{k-1} &= (I_1 \otimes I_2 \otimes \cdots \otimes A_k)\alpha_k, \\ \alpha_k &= G\eta. \end{aligned} \tag{9}$$

Stacking (9) into supervectors with $\eta = ((\mathrm{x}-\mu)', \alpha_1', \ldots, \alpha_{k-1}', \alpha_k')'$ and $\xi = (\zeta'\eta')'$ and placing the matrices of (9) into appropriate sections of β and γ of (1), shows that (1) can handle the model (7). Thus, the covariance structure (8) follows, which, being a special case of (2), can be handled by EQS.

The structures of all standard three-mode models are destroyed by variable rescaling. For example, if $x^* = Dx$ where D is a diagonal matrix, with

$$x^* = \mu^* + D(A \otimes B)G\eta + \zeta^*,$$

only μ^* and ζ^* are scale-free (i.e., have $\mu^* = D\mu$ and $\zeta^* = D\zeta$), and the Kronecker product structure is destroyed. Statistically speaking, it is thus improper to analyze arbitrarily scaled (e.g. correlation-metric) data since conclusions about the structure will depend on the metric chosen. It is possible to generate scale-invariant models. For example, Lee and Fong [10] considered the model

$$x = \mu + D[(A \otimes B)G\eta + \zeta] \tag{10}$$

whose structure in the brackets is invariant to rescaling by diagonal matrices, since the diagonal matrix D absorbs the scaling. Thus, the covariance structure

$$\Sigma = D[(A \otimes B)G\Phi G'(A \otimes B)' + Z]D = DPD \tag{11}$$

has an invariant three-mode representation under such constraints as $\mathrm{Diag}(P) = I$ or $Z = I$. The former represents the three-mode structure in a correlation metric, while the latter is in the form of Bentler's [1] factor analytic model. In similar fashion as above, it can be shown that this model is a special case of the Bentler–Weeks model with structural equations

$$\begin{aligned} &x = \mu + Dy, \qquad y = (A \otimes I)\alpha_1 + \zeta, \\ &\alpha_1 = (I \otimes B)\alpha_2, \qquad \alpha_2 = G\eta \end{aligned}, \tag{12}$$

and with same covariance structure (11).

In all the above models, the unobserved random variables η and ζ have been considered as noncausally related. Next, we consider models that allow the variables to be structurally related. For example, if

$$x = \mu + D[(A \otimes B)G\eta + \zeta] \tag{13}$$

and

$$\eta = \beta\eta + \gamma\xi, \tag{14}$$

then, if the inverse exists, $\eta = (\mathrm{I} - \beta)^{-1}\gamma\xi$. Thus,

$$x = \mu + D\left[(A \otimes B)G(I - \beta)^{-1}\gamma\xi + \zeta\right], \tag{15}$$

with the obvious covariance structure under the typical assumption that $E(\xi\zeta') = 0$. Such models generalize multimode structural models as they are currently conceived. The Bentler–Weeks representation of (15) is given by

$$\begin{aligned} &x = \mu + Dy, \qquad y = (A \otimes I)\alpha_1 + \zeta, \\ &\alpha_1 = (I \otimes B)\alpha_2, \qquad \alpha_2 = G\alpha_3, \qquad \alpha_3 = \beta\alpha_3 + \alpha_4, \qquad \alpha_4 = \gamma\xi. \end{aligned} \tag{16}$$

Substitution of (16) to yield (15) requires noting that the equation for α_3 can be equivalently written as $\alpha_3 = (\mathrm{I} - \beta)^{-1}\alpha_4$.

The models defined by (7), (10) and (15) are extensions of the basic Bentler–Lee model (4) with various special features. The following model is a general one which combines all three extensions together:

$$x = \mu + D\left[(A_1 \otimes \cdots \otimes A_k)G(I - \beta)^{-1}\gamma\xi + \zeta\right], \tag{17}$$

with the appropriate covariance structure. Again this general model can be written as a special case of the Bentler–Weeks model by defining

$$\begin{aligned} &x = \mu + Dy, \\ &y = (A_1 \otimes I_2 \otimes \cdots \otimes I_k)\alpha_1 + \zeta, \\ &\alpha_1 = (I_1 \otimes A_2 \otimes \cdots \otimes I_k)\alpha_2, \\ &\quad\vdots \\ &\alpha_{k-1} = (I_1 \otimes I_2 \otimes \cdots \otimes A_k)\alpha_k, \\ &\alpha_k = G\rho_1, \\ &\rho_1 = \beta\rho_1 + \rho_2, \\ &\rho_2 = \gamma\xi, \end{aligned} \tag{18}$$

with additional random vectors ρ_1 and ρ_2. Hence, this complicated multimode model can be handled easily via the EQS program.

4. Example

To provide an example to illustrate the materials discussed, we reanalyzed the data reported in Bentler and Lee [3] via the EQS program, which consisted of four personality traits measured by three methods of measurement. The traits were extraversion (E), anxiety (A), impulsivity (I), and achievement motivation (M), and the methods were peer report (P), teacher rating (T), and self-rating (S). Thus there were 12 measured variables (EP, AP, IP, MP, ET, AT, IT, MT, ES, AS, IS, MS), which, in EQS as shown in Appendix 1, are labeled V1–V12. The covariance matrix was reported in Bentler and Lee (actually, a correlation matrix with standard deviations unavailable), and is reproduced in the last section of Appendix 1. The proposed 3-mode model was given by (4) and (5) with $A = [I_A, a]'$, $B = [I_B, b]'$ where $I_A(2 \times 2)$ and $I_B(3 \times 3)$ are fixed identity matrices, $a(2 \times 1)$ and $b(3 \times 1)$ are parameter vectors, $\Phi(6 \times 6)$ is a fixed identity matrix, the upper triangular elements of $G(6 \times 5)$ are fixed at zero while the remaining elements of G are free parameters, and Z is diagonal with $Z(1, 1)$ and $Z(5, 5)$ fixed at zero (for correspondence to [3]). The 3×2 matrix A provides a 2-factor representation of the three methods. The 4×3 matrix B provides a 3-factor representation of the four traits. The identity submatrices of these matrices were fixed for purposes of identification. The G matrix provides the transformation of the $2 \times 3 = 6$ trait-method factors into 5 individual difference person factors.

The program code for using EQS to get the GLS solution is presented in Appendix 1. The first 14 lines of EQS code provide identifying information for the run and indicate that GLS estimation is to be used. The model as given in (6) contains three sets of equations. The first, $x - \mu = (A \otimes I)\alpha_1 + \zeta_1$ provides 12 equations for the 12 measured variables in terms of the 8 α_1 and 12 ζ factors. Thus $(A \otimes I)$ is 12×8 with only a few unknown entries. In the /EQUATIONS section of Appendix 1, the first 12 equations for V1–V12 provide the specification. The 8 α_1 factors are called F1–F8 and the 12 ζ factors are called E1–E12 in the job setup. The * indicates a free parameter and the number next to it is an initial estimate of its value. Since $(A \otimes I)$ contains equalities, these are specified in the first two lines of the /CONSTRAINTS section. A free parameter can be located by the equation involved and the predictor variable, e.g., (V9, F1) refers to the V9 predicted by F1, that is, the .5* in the equation for V9. Although equations for V1–V12 seem to contain 8 free parameters, the two constraint lines verify that there are only 2 free parameters – the number specified in a (2×1) as noted above – with the rest being functionally dependent, specifically, identical to the 2 free parameters.

The second set of equations in (6), $\alpha_1 = (I \otimes B)\alpha_2$, is given in Appendix 1 by the 8 equations for F1–F8. Since $(I \otimes B)$ is 8×6, α_2 is 6×1 with the labels $F9$–$F14$ in the program. Only 6 elements of $(I \otimes B)$ are free parameters and,

Table 1
GLS solution of Bentler and Lee's example

A			B		Z diag
1	0	1	0	0	0
0	1	0	1	0	0.20 (0.08)
0.34 (0.11)	0.25 (0.11)	0	0	1	0.15 (0.10)
		0.62 (0.13)	0.92 (0.15)	−0.55 (0.16)	0.20 (0.08)
					0.0
					0.22 (0.08)
			G		
0.88 (0.09)					0.32 (0.09)
−0.47 (0.12)	0.69 (0.10)				0.17 (0.07)
0.36 (0.12)	−0.09 (0.14)	0.71 (0.11)			0.48 (0.10)
0.56 (0.11)	0.14 (0.11)	0.09 (0.11)	0.71 (0.07)		0.62 (0.13)
−0.30 (0.12)	0.58 (0.11)	0.07 (0.10)	−0.19 (0.09)	0.44 (0.10)	0.37 (0.10)
0.33 (0.11)	0.10 (0.13)	0.55 (0.11)	0.32 (0.09)	−0.08 (0.12)	0.33 (0.08)

Note: Standard errors are in parentheses.

according to the specification, 3 of these are free and the other 3 are equal as shown in the last three lines of the /CONSTRAINTS section. The final set of equation in (6) $\alpha_2 = G\eta$, is given by the equations for F9–F14 in the /EQUATIONS stream. The 5 person factors $\eta_1 - \eta_5$ are called F15–F19 in the program.

The only variances that need estimation are those for the ζ variables, since the η variables are specified to be uncorrelated, unit variance. The free ζ variances are given in the first row of the /VARIANCES section, while the fixed ζ and

Table 2
ML solution of Bentler and Lee's example

A			B		Z diag
1	0	1	0	0	0
0	1	0	1	0	0.43 (0.09)
0.22 (0.10)	0.43 (0.11)	0	0	1	0.23 (0.11)
		0.56 (0.11)	1.13 (0.13)	−0.64 (0.15)	0.24 (0.09)
					0
					0.36 (0.09)
			G		
1.0 (0.09)					0.39 (0.09)
−0.43 (0.11)	0.59 (0.09)				0.22 (0.08)
0.45 (0.12)	−0.07 (0.16)	0.77 (0.11)			0.72 (0.12)
0.64 (0.11)	0.16 (0.11)	0.01 (0.11)	0.75 (0.07)		0.96 (0.17)
−0.28 (0.11)	0.64 (0.11)	0.03 (0.11)	−0.11 (0.09)	0.32 (0.14)	0.72 (0.13)
0.41 (0.11)	0.18 (0.13)	0.49 (0.12)	0.37 (0.09)	−0.10 (0.15)	0.55 (0.10)

Note: Standard errors are in parentheses.

fixed η variances are specified in the subsequent two lines. The input matrix completes the EQS specification.

The program converged in 5 iterations, yielding a $\chi^2_{(43)} = 52.3$. The estimates and their standard error estimates are reported in Table 1. We observe that this solution is very close to that given in Bentler and Lee [3]; the maximum difference in estimates and standard errors is .01. The same result was obtained when residual standard deviations, rather than variances, were estimated, to duplicate the result given in [3] (output not shown to conserve space). In addition, as was mentioned above, the EQS program can produce maximum likelihood estimates as well. To do this, we only need to change the input 'ME = GLS' to 'ME = ML'. The ML solution for this example is reported in Table 2. We observe that the ML solution is quite similar to the GLS solution in terms of relative size and signs of estimates. The lack of a closer correspondence is probably due to the small sample size ($N = 68$) of the data.

We have also analyzed several other data sets, and verified that the EQS program produces reasonable results.

5. Discussion

We have shown how complex multimode models can be handled by the BMDP program EQS because of the simplicity of verifying the correspondence between multimode models and the Bentler–Weeks equations, as well as the ease of implementation as shown in Appendix 1. Other programs could certainly be used as well, at least under the normal distribution assumption (no other publically distributed packages currently produce elliptical and distribution-free estimates and tests as well).

At present, perhaps the most widely used package program in covariance structure analysis is the LISREL program developed by Jöreskog and Sörbom [6]. The basic model is represented by the following structural equations

$$\eta^* = B^*\eta^* + \Gamma^*\xi^* + \zeta^*, \qquad y^* = \mu_y + \Omega_y^*\eta^* + \varepsilon^*,$$
$$x^* = \mu_x + \Omega_x^*\xi^* + \delta^*, \tag{19}$$

where $(I - B^*)$ is assumed to be nonsingular. Then from (6) we see that if we let $y^* = x$, $\mu_y = \mu$, $\eta^* = (\alpha_1', \alpha_2')'$, $\xi^* = \eta$, $\zeta^* = 0$, $\varepsilon^* = \zeta$, $\mu_x = \delta^* = 0$, $\Omega_x^* = 0$, B^* be the 2×2 supermatrix with rows $(0, I \otimes B)$ and $(0, 0)$, Γ^* be the 2×1 supermatrix $(0, G')'$, and Ω_y^* be the 1×2 supermatrix $(A \otimes I, 0)$ with $\text{cov}(\varepsilon^*) = Z$, and $\text{cov}(\xi^*) = \Phi$, then the LISREL model can handle the basic three-mode model as well. By similar reasoning as above, it can be shown that the LISREL model can be applied to obtain solutions for other more general 3-mode models as described in the previous sections. To illustrate, the job setup for the LISREL run corresponding to the ML solution of Table 2 is shown in Appendix 2. To conserve space, the input correlation matrix, the labels, and the value cards specifying start values are not included. The LISREL results agree with the EQS solution of Table 2.

This paper has emphasized the confirmatory, hypothesis testing, approach to multimode models which permits the statistical evaluation of the adequacy of a model, its parameters, and its restrictions. Of course, in many contexts, exploratory data analysis including multimode model evaluation may be more desirable. In that case, a least squares approach to model fitting, with its minimal assumptions, may be more appropriate. This can be done with the EQS program as well. An advantage of exploratory versions of multimode models is that identification conditions that are initially imposed to yield a unique solution may be modified subsequently by transformation of relevant matrices into a more interpretable form (see, e.g., [3]). Such rotations cannot be carried out within EQS or LISREL. If a component rather than latent variable type of solution is desired, more specialized programs (e.g., [8]) would probably be more appropriate.

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Appendix 1

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/TITLE
BENTLER AND LEE'S DATA (1979)
INTERCORRELATION OF FOUR PERSONALITY VARIABLES MEASURED BY PEER,
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TEACHER, AND SELF-RATINGS
/SPECIFICATIONS
CASES=68; VARIABLES=12; ME=GLS;
/LABELS
V1=EP; V2=AP; V3=IP; V4=MP;
V5=ET; V6=AT; V7=IT; V8=MT;
V9=ES; V10=AS; V11=IS; V12=MS;
F1=ALPHA11; F2=ALPHA12; F3=ALPHA13; F4=ALPHA14;
F5=ALPHA15; F6=ALPHA16; F7=ALPHA17; F8=ALPHA18;
F9=ALPHA21; F10=ALPHA22; F11=ALPHA23; F12=ALPHA24;
F13=ALPHA25; F14=ALPHA26;
F15=ETA1; F16=ETA2; F17=ETA3; F18=ETA4; F19=ETA5;
/EQUATIONS
V1=F1+E1;
V2=F2+E2;
V3=F3+E3;
V4=F4+E4;
V5=F5+E5;
V6=F6+E6;
V7=F7+E7;
V8=F8+E8;
V9=.5*F1+.3*F5+E9;
V10=.5*F2+.3*F6+E10;
V11=.5*F3+.3*F7+E11;
V12=.5*F4+.3*F8+E12;
F1=F9;
F2=F10;
F3=F11;
F4=.5*F9+.8*F10-.5*F11;
F5=F12;
F6=F13;
F7=F14;
F8=.5*F12+.8*F13-.5*F14;
F9=.8*F15;
F10=-.5*F15+.6*F16;
F11=.3*F15-.1*F16+.7*F17;
F12=.5*F15+.1*F16+.1*F17+.7*F18;
F13=-.3*F15+.5*F16+.1*F17-.2*F18+.5*F19;
F14=.3*F15+.1*F16+.5*F17+.3*F18-.1*F19;
/CONSTRAINTS
(V9,F1)=(V10,F2)=(V11,F3)=(V12,F4);
(V9,F5)=(V10,F6)=(V11,F7)=(V12,F8);
(F4,F9)=(F8,F12);
(F4,F10)=(F8,F13);
(F4,F11)=(F8,F14);
/VARIANCES
E2 TO E12 X E5 = .5*;
E1=0; E5=0;
F15 TO F19=1.0;
/MATRIX
1.0
-.38 1.0
 .42 -.21 1.0
```

```
-.25  .54 -.54 1.0
 .64 -.15  .26 -.05 1.0
-.29  .66 -.19  .44 -.25 1.0
 .38 -.09  .56 -.19  .59 -.14 1.0
-.22  .51 -.33  .66  .06  .62 -.05 1.0
 .45 -.05  .12  .10  .50 -.05  .36  .17 1.0
 .04  .38 -.03  .14  .08  .30  .09  .16  .02 1.0
 .33 -.13  .35 -.18  .41 -.14  .45 -.13  .43  .16 1.0
-.21  .37 -.44  .58 -.01  .41 -.10  .62  .06  .04 -.37 1.0
/END
```

Appendix 2

```
Bentler and Lee's data (1979)
DA NI=12 NO=68 MA=KM
MO NY=12 NE=14 NK=5 PH=ID,FI PS=ZE,FI BE=FU,FI TE=DI,FR
FI GA(1,1)-GA(8,5)
FI GA(9,2)-GA(9,5) GA(10,3)-GA(10,5) GA(11,4)-GA(11,5) GA(12,5)
FI TE(1,1) TE(5,5)
FR LY(9,1) LY(10,2) LY(11,3) LY(12,4)
FR LY(9,5) LY(10,6) LY(11,7) LY(12,8)
FR BE(4,9) BE(4,10) BE(4,11) BE(8,12) BE(8,13) BE(8,14)
EQUAL LY(9,1) LY(10,2) LY(11,3) LY(12,4)
EQUAL LY(9,5) LY(10,6) LY(11,7) LY(12,8)
EQUAL BE(4,9) BE(8,12)
EQUAL BE(4,10) BE(8,13)
EQUAL BE(4,11) BE(8,14)
OU ML AL
```