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REVIEWS

Henk A. Kiers. *Three-Way Methods for the Analysis of Qualitative and Quantitative Two-Way Data.* Leiden, The Netherlands: DSWO Press, 1989, ISBN: 90-6695-037-4, 185 pp. $22.00.

This book, like many in the DSWO Press M & T series, is concerned with optimal representation of multivariate data under dimension restrictions. Specifically, Kiers examines various lower dimensional representations of three-mode multivariate data. As an example of three-mode data, consider the responses of n objects (e.g., individuals) on each of k categorical variables. On the i-th variable, an object chooses or is assigned to one of m_i categories. Objects, variables, and categories constitute the three modes. Kiers examines methods having the following goals:

1. represent the relations among the variables in a low dimensional subspace of R^k,
2. represent the relations among the objects in a low dimensional subspace of R^n, and
3. represent the relations between the categories and the object coordinates obtained in Item 2.

The author did not describe his intended audience, but I suspect that three sets of readers (with nonempty intersections) will find the book to be useful. First, methodological researchers (i.e., the author's colleagues) will certainly be interested in the book. It is likely, however, that most of these individuals will have read the book by now, without the benefit of this review. The second set consists of empirical researchers who use multiple correspondence analysis (MCA) and related techniques to understand their data. For them, this book offers some new and useful twists on familiar methods. The third set consists of students and others who are studying MCA and related techniques. This book is too narrow to serve as the primary text in a first course on MCA, but it could be used to supplement a less specialized text (e.g., Greenacre, 1984). A working knowledge of basic linear algebra, including the vec operator and Kronecker multiplication, is sufficient for understanding the book. Calculus is not employed, although in at least one place, it would have been useful. The conjecture on page 46 is readily proven using Lagrange multipliers.

The book is carefully organized into three parts. Part I reviews and classifies three-way methods applied to quantification matrices (I'll say more about quantification matrices below). Part II describes the theory and application of two methods proposed by the author: INDOQUAL and INDOMIX. Part III illustrates INDOQUAL and INDOMIX on seven data sets and briefly discusses cross-validation and jackknife methods for assessing stability of the solutions.

The hierarchical classification in Part I is a bit forced, but it does provide a framework for comparing the author's methods to others that are better known. A paper based on Kiers's classification has appeared in this journal (Kiers, 1991a). Much of Parts I and II is devoted to summarizing methods and results that have appeared elsewhere in the literature. The summaries are terse but still interesting and valuable, especially to those (like myself) who are unable to read the original French papers (20% of the references are French publications). The examples in Part III are quite illuminating; I especially liked the cetacea example. The discussion on assessing the stability of the solutions is adequate, but I think it deserves more attention. The nonlinear nature of the estimators suggests that bootstrap estimates of standard errors (Efron, 1982) may be better than jackknife estimates (at the expense of additional computation). Exact small sample results would likely be difficult to obtain, although some progress could possibly be made under independent multinomial sampling. For certain three-mode analyses under independent normal sampling, Boik and Marasinghe (1989) obtained exact small sample results that are useful in hypothesis testing.

The book contains useful subject and author indices, a glossary of notation, and a glossary of acronyms. The latter is particularly helpful because an acronym epidemic appears to have broken out in this research area. I counted over 20 acronyms (AFM, IDIOSCAL, INDORT, INDSCAL, etc.); each referring to a specific method or variation thereof. One reason for the outbreak of acronyms (Kiers cannot be held solely responsible for this) is that an acronym such as INDOQUAL refers to an implicit model, a set of restrictions, a loss function for fitting the model, and a data type. A method based on the same model, restrictions, and loss function but used on a different data type gets its own acronym. The epidemic notwithstanding, I would add one more acronym (SVD for singular value decomposition) to the glossary. The author's use of PCA (principal components analysis) when SVD is more appropriate is a minor irritation (e.g., Equation 2, p. 10).

In Part II, Kiers gives some new results on INDOQUAL and INDOMIX. To describe these results and to allow me to comment on some technical aspects of the book, it is necessary to give a more detailed description of his methods.

The two methods proposed by Kiers are variations of Carroll and Chang's (1970) INDSCAL applied to specific quantification matrices. A quantification matrix is an $n \times n$ matrix that summarizes (i.e., quantifies) all pairwise associations among n objects on a single qualitative or quantitative variable. For a qualitative variable (say variable i), Kiers generally employed the quantification matrix

$$\mathbf{S}_i = \mathbf{P}\mathbf{G}_i(\mathbf{G}_i'\mathbf{G}_i)^{-1}\mathbf{G}_i'\mathbf{P}, \tag{1}$$

where $\mathbf{G}_i$ is an $n \times m_i$ indicator matrix giving the responses of the n objects on m_i categories, $\mathbf{P}$ is the $n \times n$ centering matrix $\mathbf{P} = \mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n'$, and $\mathbf{1}_n$ is an n-vector of ones. For a quantitative variable, Kiers employed the quantification matrix

$$\mathbf{S}_i = \frac{\mathbf{P}\mathbf{y}_i\mathbf{y}_i'\mathbf{P}}{\mathbf{y}_i'\mathbf{P}\mathbf{y}_i}, \tag{2}$$

where $\mathbf{y}_i$ is the n-vector of quantitative responses on variable i. For convenience, m_i is taken to be 1.

Kiers's INDOQUAL and INDOMIX methods represent quantification matrices for a set of k variables as

$$\mathbf{S}_i = \mathbf{X}\mathbf{W}_i\mathbf{X}' + \mathbf{E}_i, \text{ for } i = 1, \ldots, k, \tag{3}$$

where $\mathbf{X}$ is an $n \times r$ orthonormal matrix, $\mathbf{W}_i$ is an $r \times r$ diagonal matrix, and $\mathbf{E}_i$ is a matrix of residuals. The r columns of $\mathbf{X}$ give object coordinates in an r-dimensional subspace of R^n. The diagonal entries of $\mathbf{W}_i$ are nonnegative and give the loadings of the i-th variable on the r components. The matrix $\mathbf{X}$ can be considered as a random matrix or as a matrix of fixed parameters, depending on whether or not the n objects are a sample from a larger population of interest. In either case, the diagonal entries of $\mathbf{W}_i$ are fixed parameters. For specified r, the unknown matrices $\mathbf{X}$ and $\mathbf{W}_i$ for $i = 1, \ldots, k$ are fit by minimizing the least squares loss function

$$SSE(\mathbf{X}, \mathbf{W}_1, \ldots, \mathbf{W}_k | \mathbf{S}_1, \ldots, \mathbf{S}_k) = \sum_{i=1}^{k} \|\mathbf{S}_i - \mathbf{X}\mathbf{W}_i\mathbf{X}'\|^2. \tag{4}$$

The number of components, r, is chosen to be as small as possible without compromising the overall fit. When all variables are qualitative, Kiers calls the method INDOQUAL. When some variables are qualitative and some are quantitative, Kiers calls the method INDOMIX. Model (3) is structurally identical to Flury's (1988) common principal components model. In Flury's model, however, the $\mathbf{S}_i$ for $i = 1, \ldots, k$ are covariance matrices having independent Wishart distributions. Also, Flury uses an MLE loss function.

In Part II, Kiers shows that INDOQUAL and INDOMIX solutions optimize the quartimax criterion over all semiorthogonal object coordinate matrices $\mathbf{X}$. This is a keen observation and explains why INDOQUAL and INDOMIX solutions generally have a simpler structure than rotated MCA solutions. These observations also appear in Kiers (1991b). Also in Part II, Kiers modifies an algorithm due to ten Berge, Knol, and Kiers (1988) to reduce its storage requirements. Kiers's modification is based on the recognition that SSE in (4) depends on the data only through aggregate statistics. The algorithm is also described in Kiers (1989).

Kiers use of the quantitative matrices in (1) and (2) is well reasoned. Nevertheless, there is a property of these matrices that is not reflected in (3) or (4). Specifically, the quantitative matrices in (1) and (2) are idempotent (i.e., $\mathbf{S}_i^2 = \mathbf{S}_i$). That $\mathbf{S}_i$ in (2) is idempotent is obvious because $\mathbf{P}$ is idempotent. Idempotency of $\mathbf{S}_i$ in (1) can be established by showing that the diagonal matrix $(\mathbf{G}_i'\mathbf{G}_i)^{-1}$ is a generalized inverse of $\mathbf{G}_i'\mathbf{P}\mathbf{G}_i$. For $\mathbf{S}_i$ in (1), it follows that (a) rank $(\mathbf{S}_i)$ = trace $(\mathbf{S}_i) = m_i - 1$; (b) the $m_i - 1$ nonzero eigenvalues of $\mathbf{S}_i$ are each unity; and (c) $\mathbf{S}_i$ can be written in diagonal form as

$$\mathbf{S}_i = \mathbf{V}_i\mathbf{V}_i', \tag{5}$$

where $\mathbf{V}_i$ is an $n \times m_i - 1$ matrix satisfying $\mathbf{V}_i'\mathbf{V}_i = \mathbf{I}_{m_{i-1}}$. In contrast to (5), model (3) approximates $\mathbf{S}_i$ by a rank r matrix whose eigenvalues (diagonal entries of $\mathbf{W}_i$) are bounded above by one. Failure to incorporate the known structure of the quantification matrices into the model and/or the loss function weakens the appeal of the method.

The goodness of fit measure employed by Kiers is called the proportion of inertia accounted for and can be written as

$$IAF_I = 1 - \frac{SSE_r}{SSE_0},$$

where SSE_r is the minimized SSE in (4) and SSE_0 is SSE when $r = 0$. From the idempotency of the quantification matrices, it follows that IAF_I is not a scale free measure of goodness of fit. Specifically, the proportion of inertia accounted for by an r dimensional model satisfies

$$0 \leq IAF_I \leq \frac{k + \sum_{i=1}^{k} \delta_i [rI_{[0,m_i-1]}(r) + (m_i - 1)I_{[m_i,\infty]}(r) - 1]}{k + \sum_{i=1}^{k} \delta_i (m_i - 2)}, \tag{6}$$

where $\delta_i = 1$ if the i-th variable is qualitative, $\delta_i = 0$ if the i-th variable is quantitative, and $I_{[\,.\,]}$ is the indicator function. For example, IAF_I values corresponding to a set of variables with category structures like those in the cetacea example are bounded above by 0.35, 0.63, 0.86, and 1.0 for $r = 1, 2, 3$, and 4, respectively. Clearly, IAF_I values must be interpreted carefully.

In sum, Kiers has written an informative book that describes recent research and provides a few new results. The publication of these results as journal articles (Kiers, 1989, 1991a, 1991b) does not really detract from the book's usefulness; it is handy to have the results nicely organized. The INDOQUAL and INDOMIX methods advocated by Kiers are quite promising, but some aspects of these methods require additional work.

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