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A user-oriented overview of multiway methods and software

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Abstract: This paper provides a brief review of the most widely known methods and software dealing with multiway data. The main features are described focusing on applicative capabilities, in order to make the choices of users easier.

Keywords: Multiway analysis; Computer programs

1. Introduction

The availability of information concerning software which carries out multiway analyses is a necessary condition for the widespread usage of multiway methods. In this CSDA special issue some of the most widely known computer programs are presented by the Authors, taking into account the following aspects: characteristics of input data, data manipulation, mathematical models, optimization algorithms, applications and other practical information.

The main purpose of this paper is to provide a brief presentation of multiway methods and software, which are well known and tested in many applications, including also some programs not presented by the Authors in this volume. However, we apologize for any omission occurring in this review.

Emphasis will be given to application capabilities rather than theoretical foundations, therefore the methods are classified according to the data they can manage. A division in three groups is considered in the following three sections: methods for the analysis of quantitative data, those for categorical or mixed data and others for proximity or preference data. Of course, the methods dealing

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with different measurement levels or with different kinds of data cause some overlaps between the groups. However, in spite of some repetitions, this organization of the paper should make the program choices of the users easier and clarify the role of the software in the multiway framework.

In the following we shall deal more frequently with methods based on purely geometrical rather than probabilistic models, consequently when referring to data matrices in general, terms like variables and objects (or units) will be adopted instead of the more specific random variables and sample units definitions.

2. Analysis of quantitative data

Data considered in this section are classified in three ways pertaining to three different classes of entities (modes) that, unless differently indicated, will be objects (or units), quantitative variables and occasions (or conditions, sets, replications, etc.). According to the definitions given in Coppi (this volume) two types of three-mode data will be distinguished: three-way data sets (different sets of variables are observed on the same objects or, conversely, the same variables are observed on different sets of objects) and three-way data arrays (same objects and same variables observed on each occasion). Before reviewing some methods dealing with the two previous kinds of data, we point out that several types of two-mode three-way data can be derived in both the cases mentioned: for instance a set of cross-product matrices between the variables (or proximity matrices between the objects) might be computed from a three-way array. Then the derived data could be analysed by the methods considered in section 4.

Three-way data sets

Two or more sets of quantitative variables can be observed on the same objects for different purposes. A few examples could help to clarify this point.

For instance, several batteries of psychological tests can be administered to a group of students to study learning abilities and their relationships. Besides a measure of the correlation between sets of tests one might be interested in finding out a few latent variables summarizing those relationships.

Typical sensory three-way data sets concern a group of instrumental variables assumed as predictors of a set of sensory variables, both measured on the same set of products. Emphasis will be given in this case to linear combinations in the former group that "best predict" variables in the latter.

Furthermore, in some economic applications the degree of development in several areas might be measured by adopting more sets of indexes. Possible homogeneous typologies of the areas might be obtained by comparing the similarity structures pertaining to the different sets.

It turns out from the previous examples that different purposes (correlations, predictions, typologies, etc.) can be achieved by focusing on different aspects of the information conveyed by multiway data (see also Coppi in this volume for a more general discussion of this aspect). Consequently, the choice of the methods to analyse the data should reflect the peculiar type of information dealt with.

Canonical Analysis is one of the first approaches proposed to analyse three-way data sets. Historically the case of two sets (Canonical Correlation Analysis, CCA) was first considered by Hotelling (1936) and afterwards extended to multiple sets (Generalized Canonical Analysis, GCA) by several authors.

The main purpose of the analysis is the quantification and description of dependence among sets of variables which usually, differently from other techniques such as Regression or Discriminant Analysis, play a symmetric role. For instance, in the psychological example given above two sets of tests could be administered to a student population to study the correlation among logical and practical abilities.

Two new variables (canonical variables), one for each set, are computed as linear combinations of the original variables, in such a way as to maximize their correlation coefficient. The two canonical variables represent the two original sets, and their correlation coefficient is a measure of the correlation between those sets. The process can be repeated to determine new pairs of canonical variables, maximally correlated between them and uncorrelated with the previous ones.

It is assumed that each pair represents a particular aspect of the "general correlation" between the two sets of the original variables. From a geometrical point of view the process can be described as the determination of "suitable" subspaces of the two linear spaces spanned by the groups of original variables (Cailliez and Pages, 1976).

Many general purpose statistical packages contain routines for performing CCA (e.g. BMDP, SAS, SPSS-X). Classical multivariate tests for canonical correlations and graphical representations of objects and variables using the first canonical variables are obtainable. Relationships among the variables and proximities of the objects can be described according to the interpretation of the canonical variables.

One theoretical interest of CCA is that it constitutes a general framework which unifies a large number of techniques if the presence of qualitative variables in the groups is allowed (Cailliez and Pages 1976, Gittins 1985). On the other hand, the assumption of the symmetric role of the sets of variables seldom occurs in practice. Going back to the sensory example given above, it is evident that one is interested in how well the sensory qualities (criterion set) can be predicted from the instrumental variables while explaining a maximum of variance of the criterion set (see van der Burg and Dijksterhuis 1992, for this type of application). Let us suppose that a few linear combinations of the instrumental variables may predict the criterion variables quite well, then the analysis will focus on subspaces of low rank contained in the space spanned by the predictors. Rao (1964), Stewart and Love (1968) and Van den Wollenberg

(1977) proposed asymmetric versions of Canonical Analysis (Redundancy Analysis) to find opportune subspaces. A program for this asymmetric approach in the case of two sets is implemented in the SAS package for instance (see options in CANCORR procedure).

GCA represents a multi-set extension of the CCA which was developed after the sixties by Horst (1961), Carroll (1968) and others. Many contributions have been added on these lines, and general reviews can be found in Kettenring (1971), Van de Geer (1984) and Gittins (1985).

A set of canonical variables for each group of the original variables and a set of "auxiliary variables" are determined by maximizing a measure of their relationship (usually equivalent to a function of the correlation matrix among the canonical variables). The auxiliary variables, which represent different aspects of the "general correlation" among the groups, span a subspace of the object-space that can be considered as a "compromise" space.

Recently nonlinear generalizations of Canonical and Redundancy Analysis have been introduced in Gifi (1981, 1990), Van der Burg, de Leeuw and Verdegaal (1988) and Van der Burg and de Leeuw (1990). The nonlinear approach determines canonical variables which are combinations of nonlinear transformations of the original variables, thus the results will be invariant under the chosen type of transformations. In this case we seek the solution in a linear subspace which contains the subspace spanned by the observed variables.

Two Alternating Least Squares (ALS) programs called CANALS (Gifi 1981) and OVERALS (Verdegaal 1986) were produced to incorporate respectively the nonlinear generalizations for two and multiple groups of variables. The characteristics of the program OVERALS along with applications to real data are described in this volume by Van der Burg et al..

When the definition of typologies of the objects is the main objective of the analysis (as for the economical example given before), or everytime we are interested in a global strategy involving separate and joint analyses of the three modes in an explicit manner, a useful class of methods is represented by the techniques following the "Interstructure-Compromise-Intrastructure" (ICI approach; Glacon 1981, p. 22), described by Coppi (this volume). In this case three different stages of analysis are emphasized.

The first stage, called interstructure analysis, concerns the study of the relationships between the groups of variables from a global point of view; the quantification of the "general correlation" between the pairs of groups can be a result of the method or can be explicitly defined as a measure for the comparison of data matrices (usually a sum of covariances or correlations). A graph in which the groups of variables are each represented by a point is provided, based on the selected type of quantification.

The second stage, called the compromise analysis, concerns a study of the proximities between the objects, based on the weighted means of the proximities associated with each group of variables. In this case, a configuration in which each object is represented by a point, displays the chosen mean proximities.

The third stage, named the intrastructure analysis, concerns an analytical study of the relationships among the variables and the proximities among the objects in the different groups of variables. The analysis is carried out using the configuration obtained at the second stage, on which the variables and the objects for each group are opportunely represented by appropriate projected points. When the occasions are ordered, a trajectory can be drawn for each object point making the interpretation of the configuration easier.

The second and third stages are usually based on the Principal Component Analysis (PCA) of a matrix obtained by weighting and juxtaposing the data matrices associated with each group of variables. The weights are usually estimated before the application of the PCA using different kinds of procedures.

In this framework we can consider two important methods: the first proposed by Escoufier (1973, 1977) and the second by Escofier and Pages (1984). The latter also can be considered in the framework of GCA by defining the fit in a suitable way. A detailed description of the two methods and the corresponding programs STATIS and AFMULT is given by Lavit et al. and Escofier et al. in this volume.

It is useful at this point to make some remarks on the differences between the methods based respectively on the Canonical Analysis and the ICI approach.

Graphical representations similar to the graph of the interstructure analysis are seldom considered when one uses GCA, thus the comparison essentially refers to the results obtainable in the compromise and the intrastructure analysis. The greatest difference from this point of view lies in the "concept of compromise" underlying the two approaches, the intrastructure usually being recovered, for each group of variables, by projections of objects and variables onto the compromise space.

The compromise determined using GCA is much more influenced by the organization of the variables in groups than by the methods based on the ICI approach. In the first case the axes of the space (the auxiliary variables) are determined in order to maximize functions of the multiple correlations with the sets of canonical variables, whereas in the second case the "explained inertia" of the variables in the direction of the components of the compromise has to be maximized.

Consequently the application of GCA seems more justified when the main objective of the analysis is the study of the relationships among the sets of variables and the objects are analysed from this point of view. On the other hand the ICI methods should be preferred when the relationships "within" the groups, in terms of inertia, are also interesting. In this case objects and variables both have the same importance in the analysis.

The problem of the determination of a "compromise" and an "intrastructure" for the objects can be handled as a problem of comparison of separately obtained configurations, using Generalized Procrustes Analysis (GPA, Gower 1975). Suppose we have one matrix of object coordinates for each group of variables, obtained using the same or different ordination techniques. For instance, different researchers have collected data on the same individuals and applied multivariate methods to produce several multidimensional representations. The main question arising in this case is, to what extent do the different sets of points contain similar information. GPA allows geometrical comparisons transforming all the configurations at the same time, so that each set of points matches the others as closely as possible, according to an opportune loss function (usually a sum of the distances between each transformed configuration and the centroid). Good minimizations provide us with representative centroids, that can be assumed as a "compromise". The superimposition of the transformed configurations enables us to analyse the intrastructural information also, because each object is represented by a cluster of points and by a centroid lying in the middle of the cluster. The possibility to use procrustes statistics to provide representations in which each group of variables is represented by a single point is also considered in Gower (1987, p. 275).

The theory of GPA is developed in Gower (1975), Ten Berge (1977), Lingoes and Borg (1978) and, more recently, in Commandeur (1991). All the main types of Procrustean transformations are carried out by the program PINDIS (Lingoes and Borg, 1976). Some interesting results concerning the comparison between the GCA, the ICI and the GPA approaches can be found in Ten Berge (1977), Glacon (1981), Kiers (1988, 1991) and Commandeur (1991).

Some of the methods considered above can also be applied in the other case of three-way data sets in which the same variables are observed on each occasion on different objects (e.g. the same tests have been administered to subjects from two or more different groups). The extension of STATIS to the analysis of a set of variables observed on different populations is considered in Lavit et al. (this volume). GPA does not constrain the elements of the compared configurations to be objects, thus sets of variable points might be matched as well. Moreover Gower (1989) reviewed standard techniques like Discriminant Analysis, Canonical Variate Analysis and some of their interesting extensions which could be usefully applied.

Several generalizations of PCA have been proposed to deal with the same variables observed on different sets of objects. For instance, simultaneous factor analyses based on the PCA on averages of the correlation matrices across occasions were considered by Levin (1966) and, more recently, extended by Rizzi and Vichi (1992). An interesting method called Simultaneous Component Analysis has been developed by Millsap and Meredith (1988) and Kiers and Ten Berge (1989) to reveal components with common interpretation across the occasions. This is especially for cases in which separate PCA in each group are not effective. The method defines component weights such that the total amount of explained variance is a maximum. The program SCA (Kiers 1990) allows one to calculate the components by an alternating least square algorithm.

Three-way data arrays

In many empirical studies the three modes (objects, variables, occasions) pertaining to the observed data are fully crossed. For instance, longitudinal data are obtained when the observation of a group of variables is repeated on the same objects. Kroonenberg (1983) reanalysed data available for a set of hospitals measured on the same variables in consecutive years to study their different patterns or rates of growth. Data arrays are also obtained when different subjects are requested to rate a set of stimuli using the same criteria. Harshman and De Sarbo (1984) presented an example from marketing research in which automobiles and celebrities were rated by several subjects using a set of bipolar rating scales in order to decide which celebrity should be chosen as spokesman for a given automobile. Moreover Lavit et al. (this volume) analyse data concerning the judgments of some students on the teaching method of their professors.

Some of the methods for three-way data sets lose their peculiarities when applied to data arrays. The analysis of correlation between the groups performed with GCA is less interesting when the variables are the same in each group. The weighting system for the occasions adopted by the methods following the ICI approach is less relevant, especially if the occasions are time periods. However in this approach an interesting new feature of the intrastructure is the possibility to draw a trajectory for each variable, besides for each object, when the occasions are ordered. Thus the evolutions of objects and variables and their relationships are displayed in the intrastructural configuration.

Specific methods for three-way arrays proposed in the factorial tradition are usually distinguished in two groups: those based on component models and others based on common factor models (or covariance structure models). Basically, methods in the first group are mainly 'data analytic' and exploratory (the three modes are considered fixed) whereas those in the second one are more probabilistic and confirmatory (the object mode is stochastic). The parameters of the common factor models are usually estimated by fitting the derived set of covariance (or correlation) matrices instead of directly fitting the data (cf. Kruskal 1978 for a discussion of direct versus indirect fitting). For this reason they will be presented in section 4. In the following we recall some examples of methods based on component models.

New possibilities for the analysis of three-way data arrays are provided by the methods based on multilinear models which assume a symmetric point of view with respect to the three modes (see Coppi in this volume for a more formal presentation). A general quadrilinear model was proposed by Tucker (1963, 1964, 1966) as a generalization of the Singular Value Decomposition (SVD) for two-way matrices.

Tucker introduced a new factorial approach based on the idea that a different underlying structure is associated with each mode and extended the factorial tradition of a single set of factors. The three "observational modes" of the data array are each associated with a "derivational mode", that can be thought of as a set of factors or idealized categories. Therefore objects, variables and occasions are considered as, respectively, linear combinations of "idealized" objects, "latent" variables and "prototype" occasions. The relationships between the three sets of factors are taken into account in a three-way array, the "core matrix", estimated by the model.

A trilinear version of the general model was also considered by Tucker (1972), particularly for cases in which one of the three modes cannot be meaningfully reduced (e.g. time indexed data). Moreover Kaptein, Neudecker and Wansbeek (1986) extended Tucker's models to an arbitrary number of ways.

The algorithms developed by Tucker did not produce a least squares approximation to the data. Kroonenberg and De Leeuw (1980) proposed ALS algorithms to fit both the quadrilinear and the trilinear models, whose application is possible by the TUCKALS programs, described in this issue by Kroonenberg.

The difficulties concerning the interpretability of the core matrix in real applications suggested considering trilinear models based on simpler assumptions. Two interesting methods, called CANDECOMP and PARAFAC, were proposed respectively by Carroll and Chang (1970) and Harshman (1970). Both these methods are based on the same trilinear model and share the intrinsic axis property, that is they do not present the rotation problem (Kruskal 1976, 1977). Unlike Tucker's conception, only one set of factors underlying the observational modes is assumed by CANDECOMP/PARAFAC models. Important differences are the procedure to extend the model to higher way cases, implemented only by the first method, and the preprocessing procedure, particularly developed by the second method. Furthermore a version of CANDECOMP called CANDELINC was also proposed by Carroll, Pruzansky and Kruskal (1980) by which linear constraints on one or more of the parameter matrices were introduced to take into account either external information regarding the elements of the constrained modes or a specific analysis of variance (ANOVA) design.

The method CANDECOMP can be applied (up to seven way extensions) using the programs INDSCAL (Carroll and Chang 1970; Chang and Carroll 1969) and SINDSCAL (Pruzansky 1975). The program PARAFAC is described by Harshman and Lundy in this issue.

Interesting comparisons between the methods considered in the framework of three-mode data are available, for instance, in Jaffrennou (1978), Glacon (1981), Law et al. (1984), Escoufier et al. (1985), Coppi and Bolasco (1989), Kiers (1988, 1991) and Kroonenberg (1992).

3. Analysis of categorical or mixed data

We have seen (Coppi, this volume) that in the presence of qualitative data it is possible to structure the data array in different ways depending on the point of view we intend to adopt. Moreover, depending on the particular structure chosen, it is possible to apply different techniques. In this section we start with a more detailed description of multiway data structures with categorical or mixed data, then, after some general considerations of the different approaches, we indicate some software programs for each structure.

We will consider the following structures:

(1) *Object by variables table*. It could contain only qualitative variables, or qualitative and quantitative variables (mixed case), or only quantitative variables. The last case has been considered in section 2. If the variables, supposed qualitative or mixed, are grouped in several sets, we obtain a three-way data set with same-objects and different variables on each occasion. If we have one qualitative variable with K categories while the others are quantitative, we could consider this qualitative variable as defining K "occasions" in order to apply the methods of the previous section. Also in this case it is defined a three-way data set but with different objects and same variables on each occasion. If this possibility is not satisfactory, or we have more qualitative variables we should consider the methods introduced in this section.
(2) *Multiple contingency table or set of multiple contingency tables*. They can be obtained from one or more object-by-variables tables if all the variables in the tables are qualitative. Of course in this transformation to frequency data we lose the possibility to assign a label to each object. It is ambiguous whether a multiple contingency table is a real multiway data set. In fact three qualitative variables are sufficient to obtain a three-way contingency table, with the three-way fully crossed, but only when a variable assumes the role of "occasions" can there be a genuine three-way data set. When we observe the same qualitative variables on different groups of objects we obtain one objects by variables table (with the objects grouped in K sets) and we can construct a multiple contingency table. If we observe the same objects and the same variables on K different "occasions" we obtain one objects by variables table (with the variables grouped in K sets) and we can construct a set of multiple contingency tables.
(3) *Derived tables*. We use this name to indicate a set of proximity tables obtained from the previous two data-structures using an opportune index. This multiway data set could be considered, from one point of view, as more general than the others. Examples of derived tables are: the proximity tables among categories or objects, and the tables of residuals with respect to a model.

There are techniques which can be used to analyse more than one of these structures. To make the illustration of the techniques easy, we will consider an overlapping classification of the methods according to their ability to analyse the kinds of data structures described above.

It is useful to note some aspects which characterize the different approaches of the methods to the analysis of a qualitative or mixed multiway data-set.

Whatever the structure we are analysing we suppose one mode (the objects or the variables) constant across the occasions. The methods which analyse a (set of) multiple contingency table(s) require a multiway qualitative data-set with the same variables on each occasion.

The interactions of various orders among the variables are a key aspect in the method. This context is easily dealt with by the methods analysing the frequencies of a multiple contingency table while the methods of the first group, which use interactive variables or nonlinear transformations, are less effective. A growing interest on this problem has led to the proposal of a combined application of the methods of the first and second groups (see f.e. Daudin and Trecourt 1980, Cavedon et al. 1982, van der Heijden and de Leeuw 1985, Leclerc et al. 1985, Aitkin et al. 1987). Another proposal considers the application of multilinear models to the matrix of residuals obtained using a log-linear model (e.g. Kroonenberg 1983).

Another aspect to evaluate is the role we want to assign to the variables (e.g. to the sets of variables) in the analysis. The variables can have a symmetric role (as e.g. in the log-linear approach) or an asymmetric role, as for example when we want to predict one set of variables on the basis of another set (e.g. Qualitative Redundancy Analysis, Israels 1984).

If we do not introduce some simplifications (e.g. the categorization of quantitative data) the presence of mixed measurement level variables requires specific techniques. The treatment of the mixed case has received growing interest in the last 10 years. Now there are methods which are able to analyse mixed variables by introducing a new set of parameters in the structural model. The new parameters could be used to introduce a nonlinear transformation of quantitative variables or quantifications of qualitative data (see e.g. van der Burg and de Leeuw in this volume, Gifi 1990, Di Ciaccio 1988). These techniques refer to the Optimal Scaling approach or to the Breiman–Friedman approach (see Breiman and Friedman 1985, Friedman 1991, Coppi and Di Ciaccio 1993).

Techniques able to analyse multiway mixed measurement level data are considered separately at the end of the section.

Object by variables tables of qualitative data

Several methods can analyse objects by variables tables of qualitative data in a multiway approach. The following is a non-exhaustive list: Nonlinear Canonical Correlation Analysis (van der Burg et al. 1988), Qualitative and Nonlinear Redundancy Analysis (Israels 1987, Meulman 1987, Van der Burg and de Leeuw 1990), Conditional and Partial Correspondence Analysis (Escofier 1988, Yanay 1986), Non-symmetrical Correspondence Analysis (D'Ambra and Lauro 1989), Canonical Correspondence Analysis (Ter Braak 1986), Multiple Factor Analysis (Escofier and Pages 1984), Weighted Additive Model (WAM, Takane et al. 1984).

Several authors pointed out that important links exist among many of them (see Israels 1987, Keller and Wansbeek 1983, Sabatier et al. 1989)

In this issue we consider two methods in particular which have had the proper software for some years: Nonlinear Canonical Analysis with the program OVERALS (now included in the package SPSS) and Multiple Factor Analysis with the program AFMULT (now also included in LADDAD package). Both these methods analyse a data matrix in which only the variables can be different on each occasion. The variables are divided in K groups and in each group we can have the same variables or different variables.

We have already considered these methods in the previous section for the analysis of quantitative variables. Now we are considering the case in which all the variables are qualitative, but we will see at the end of the section that both methods can treat mixed measurement level variables.

The OVERALS method originates from the Optimal Scaling tradition (Gifi 1990, Young 1981). In fact we can see it as a K-sets Canonical Correlation Analysis with Optimal Scaling, or a K-sets Homogeneity Analysis (Multiple Correspondence Analysis) with additivity constraints and Optimal Scaling (cfr. van der Burg, de Leeuw, Verdegaal 1988). Using Optimal Scaling and Alternating Least Squares (ALS) the OVERALS algorithm can analyse nominal (single/multiple), ordinal and numerical variables (see the end of the section for the mixed case). The program can also manage "passive variables", that is variables that are ignored by the loss function but which can be represented in the final solution. Another suggested possibility is to obtain a "partial solution", that is, we can partial out a variable, making copies of the variable in each set. Applications of the method are shown in Gifi (1990), van der Burg and Dijksterhuis (1989), Verdegaal (1986).

With the AFM method we can treat variables which have a nominal measurement level, but we cannot consider the ordinal level, as we do in OVERALS. If we have only qualitative variables, AFM can be considered as an extension of Multiple Correspondence Analysis (MCA). In fact if we have only one qualitative variable in each group, AFM and MCA are equivalent. In this particular case also OVERALS, with multiple category quantification, gives us the same results. Furthermore, if we consider also rank-one restrictions on the multiple category quantifications, OVERALS gives the same results as PRINCALS (Nonlinear Principal Component Analysis, Gifi 1990, Young et al. 1978). In the other cases AFM and OVERALS produce a different analysis, as we have already noted for quantitative variables in the previous section. In the general case with several qualitative variables in each group, AFM weights each group differently and there is not equivalence with MCA anymore. In this case we can see AFM applied to qualitative data as a weighted PCA of the indicators of variables categories. The weights can be considered as a way to "balance" the contribution of each group to the analysis. Interesting features of the program are the treatment of missing data, the possibility to weight the variables and the "aides à l'interpretation" of the results.

Some limited extension of the STATIS method to consider qualitative data are dealth with in Glacon (1981) and Escoufier (1980) but they are not implemented in the software available. We note also the program ALSCOMP3 (Sands and Young 1980) which extends the method PARAFAC to the analysis of qualitative or mixed data.

There exists also software allowing the application of more than one technique to analyse a multiway objects by variables table. In a non-commercial framework we wish to point out the program CANOCO (Ter Braak 1986), including simple and partial Canonical Correspondence Analysis, and the program LAMDA (Lauro and D'Ambra 1983) including Nonsymmetric Correspondence Analysis. Among the commercial software we note SAS package and SPSS package.

Multiple contingency tables

We distinguish between two kinds of data: (a) we observe the same qualitative variables on different groups of objects; (b) we observe the same objects and the same variables on K different "occasions".

In case (a) we have a multiple contingency table for each group of objects, but we can also obtain a unique multiple contingency table if we join the groups of objects and introduce a new dummy variable to discriminate each group. As an example of type (a) data, we could observe the same sociological data in K different periods of time, on J different groups of individuals. In this case we are interested in the comparison of the relationships among the variables on each occasion.

In case (b) we obtain a set of multiple contingency tables. We are, for example, in this situation if the same sociological data were observed in K different periods of time on the same individuals. This is a three-way array of qualitative data. In this case it would be incorrect to transform the data in a single multiple contingency table, because an object would be considered more than once in the table.

In both cases (a) and (b) we assume that the basic data are constituted by several objects by variables tables. In case (a) we can construct a single multiple contingency table only by adding one variable which assumes the special role of "occasions".

A general limit for the analysis with multiple contingency tables is that we lose the identification of the objects. This limit is more evident in case (b) in which an interesting goal could be to describe the different behaviour of the objects on the K occasions. In this sense the analysis of data (b), transformed as multiple contingency tables, cannot be considered completely satisfactory.

The analysis of a multiway contingency table has been developed particularly with a probabilistic approach. The most frequently utilized methods are the Logistic and Log-linear Model Approach (Cox 1972, Bishop et al. 1975) and the Latent Class analysis (Lazarsfel and Henry 1968, Goodman 1974). A more recent proposal is the Log-non-linear Model Approach (Goodman 1986). It explicitly introduces a (multiple) quantification of categorical variables, showing interesting links with other methods based on the scaling of the categories.

These are essentially two-way methods and have strong limits in the analysis of data (a) or (b). A three-way method is the Simultaneous Latent Structure Analysis (Clogg and Goodman 1984) which is able to make a latent structure analysis of data (a) correctly. This method can be used to check homogeneity constraints on the models parameters of each table. An analysis of a set of contingency tables using log-bilinear model was proposed by Becker and Clogg (1989). Choulakian (1988) and Mooijaart (1992) proposed the use of log-trilinear models.

These methods also have some powerful features. They have a probabilistic approach and we can obtain a reliability analysis of the results usually not obtainable with other techniques, unless using resampling methods (Bootstrap or Jackknife).

Another method that combines descriptive and confirmatory features is the analysis of contingency tables by Ideal Point Discriminant Analysis (Takane et al. 1987).

Several contributions in this field refer to a joint use of the log-linear approach with explorative methods. We note the Generalized Correspondence Analysis of Multiway Contingency Tables (van der Heijden and Meijerink 1989), or the three-way analysis of residuals from a log-linear model (Kroonenberg 1983, D'Ambra and Kiers 1991).

An important problem with multiple contingency tables analysis is that usually there are serious difficulties to analyse more than 4 or 5 variables at a time. Moreover the variables should not have too many categories. Otherwise there will be too many zeros or low frequencies in the cells of the table and the methods will not be correctly applicable.

For the analysis of multiple contingency tables several packages are available (SAS, SPSS, GENSTAT, BMDP...) which contain logistic or log-linear procedures. Other more specialized software should be requested from the authors.

Derived tables

We consider now another possibility consisting in the transformation of the previous data structures in order to obtain a set of proximity tables between objects, categories, variables or occasions. The aim of this is to obtain a rearranging of the data allowing a simpler or more powerful analysis. Depending on the particular transformation adopted, on the particular index chosen and on the techniques of analysis applied to the proximity tables, we obtain a very large number of conceivable methods. From this point of view the approach based on the derived tables can be considered more general than other approaches. In fact using proper indices and applying Three-way Multidimensional Scaling techniques, we can obtain the same analysis as some of the previous methods.

Proximity tables among categories of qualitative variables can be obtained by an association index based on odds-ratios (Di Ciaccio 1986). Using this index we can transform a set of multiple contingency tables in a set of proximity tables which can be analysed by three-way Multidimensional Scaling (Coppi 1986, 1988).

The relation measurements between pairs of objects are analysed by D'Ambra and Marchetti (1986) using a generalized PCA and they are also considered by Marchetti (1988) and Kiers (1989) combined with three-way methods, but these approaches were developed for the analysis of a two-way qualitative data set.

If we want to extract only some aspects of the information from the original data, it is appropriate to construct tables of residuals with respect to a model which explains the less interesting aspects of the data. In fact we can obtain our aim just by analysing these tables of residuals. Alternatively we could obtain tables of coefficients describing a particular aspects (for example a three-way interaction) of a model (for example the log-linear model). These kinds of tables could be analysed using methods such as, for example, the Three-mode Component Analysis (Kroonenberg 1983) or the PARAFAC (D'Ambra and Kiers 1991).

The approaches which use derived tables appear to be less 'natural' than the previous methods. This is partly due to the two-step procedure (derive tables – analyse tables) and partly due to the difficulties in explaining the meaning of the new (derived) tables introduced by these approaches.

Mixed measurement level

When we have both qualitative and quantitative variables the most convenient structure to analyse is a set of objects by variables tables. If the role of all the variables is the same in the analysis then we can apply both the AFM and OVERALS programs, already considered. The two programs include the treatment of mixed variables in quite different ways. In OVERALS, according to the Optimal Scaling approach, we only have to declare the measurement level of the variables, while in AFM we have to group the variables in such a way that in each group we have only one measurement level. This feature allows us to consider a two-way analysis with qualitative and quantitative variables, but it creates some difficulties in three-way analysis because in this case each group of variables does not correspond to one occasion anymore.

Another three-way method based on Optimal Scaling is REDUNDALS (van der Burg and de Leeuw 1990). This technique can be applied when we have two sets of variables and the aim of the analysis is the optimal prediction of one set from the other. This is a generalization of Redundancy Analysis (Van den Wollenberg 1977) to the analysis of mixed measurement level of data.

Another approach is described by Kiers (1989) that constructs a "quantification matrix" for each variable, depending its measurement level, and then applies three-way scaling to the entire set of matrices. This approach is devoted to the analysis of two way data but could be extended also to the analysis of three-way data.

4. Analysis of proximity and preference data

The choice to consider proximities (three-way two-mode) and preferences (three-way three-mode) together in this section is justified by the close relationships of the geometric and algebraic models adopted for the two types of data.

Scalar product and proximity data

Data obtained by observing the same variables on different occasions can be transformed into a set of cross products (or covariance or correlation) matrices by which to analyse the relationships between the factorial structures. The TUCKALS, CANDECOMP and PARAFAC methods can deal with this kind of derived data, moreover other important proposals were made, for instance, by Joreskog (1970, 1971), and McDonald (1978, 1984). Snyder, Law and Hattie (1984) provide a brief review of these methods whose application is usually performed by the programs LISREL (Joreskog and Sorbom 1988), COSAN (McDonald 1978, Fraser 1980) and MUTMUM (Browne 1990). In particular COSAN, which is a general purpose program for the analysis of covariances structures, allows one to carry out the proposals based on the invariant factors multimode model by McDonald (1984) and on the longitudinal factor analysis by Swaminathan (1984). The program LISREL is included in the SPSS-X package and some covariance structure analyses can be performed by the procedure CALIS of the SAS package.

A set of symmetric proximity matrices can be analysed by spatial and nonspatial models to obtain information concerning the differences between the structures of (dis)similarity and their stability. The Multidimensional Scaling (MDS) methods which handle this problem are called "individual differences" methods because of their psychological origin. Historically a first proposal to deal with sets of proximity matrices was the "points-of-view" analysis of Tucker and Messick (1963). Two stages of analysis are considered: first a matrix of correlations between pairs of data matrices is computed and factor analysed to obtain a representation of the homogeneous groups of occasions. Then a weighted average for each group is computed and represented by standard MDS techniques; each of the configuration obtained represents a different point of view. Some relationships of this method with the ICI approach can be pointed out, in fact we can consider the first and the second stage of analysis described above very close to the "interstructure" and "compromise" of the ICI approach.

Successively many authors sought a more parsimonious solution to the individual difference problem. The Carroll and Chang (1970) proposal, whose model was also considered by Bloxom (1968) and Horan (1969), represented an important innovation and the starting point for methods of multiway MDS. Their model, called INDSCAL (for INdividual Difference SCALing), assumes that a "common space" ("compromise level") exists in which each object is represented by a single point whose coordinates refer to the latent variables associated with the occasions. The "individual differences" can be considered by a set of occasion weights by which it is possible to shrink or stretch each dimension to have separate representations ("individual spaces"). A graph in which each proximity matrix is represented by a single point is obtainable by using the occasion weights as coordinates ("interstructural level"). The weighted Euclidean model is adopted and shares the important property of "uniqueness" with CANDECOMP and PARAFAC.

Many programs perform the analysis proposed by the INDSCAL model, even if differences exist concerning the fitting criterion and other features. Besides the programs INDSCAL, SINDSCAL, PARAFAC, TUCKALS2 and PINDIS, it is possible to use ALSCAL (Young and Lewyckyj 1979) and MULTISCALE

(Ramsay 1977,1982). An ALSCAL procedure is also included in the SPSS-X package.

Many generalizations of the INDSCAL model have been proposed. The idea on which they are based is to introduce a greater flexibility in the type of transformations which provide the "individual spaces" from the "common space". Some of these generalizations are represented by the Carroll and Chang (1970, 1972) IDIOSCAL model, the PARAFAC2 by Harshman (1972), the "three-mode scaling" by Tucker (1972), and other procedures proposed by Bloxom (1978), Ramsay (1981) and Young (1984). The program IDIOSCAL and many of the programs cited above can manage this type of generalizations.

An important approach to be considered in the framework of three-way proximity analysis is represented by the maximum likelihood MDS developed by Ramsay.

MULTISCALE is a multidimensional scaling program which analyzes square data matrices with a probabilistic approach. Each datum is supposed to reflect a population value plus error. The program assumes independent Normal (or Log-Normal) distributed errors and fits the data by Maximum Likelihood Estimation (MLE). It provides confidence regions for each parameter and tests the significance of the improvement of fit as we move from a simpler to a more general model. Of course, the desirable properties of MLE are not relevant if the error assumptions are violated, thus the program provide diagnostic plots for error control. The program can fit three kinds of models: Euclidean, Weighted Euclidean (INDSCAL) and Full Metric (IDIOSCAL).

A comparison between MULTISCALE, ALSCAL, INDSCAL and other MDS programs is made in Shiffman et al. (1981) where it is noted that the solutions are similar when the data are not exceedingly noisy. The different approaches to fit the INDSCAL model by several MDS programs are considered in detail by Arabie et al. (1987).

Proximity data can also be represented by nonspatial models like trees, whose main purpose is to cluster the observed objects. For the two-way case a general formalization based on the concept of matching function was provided in Tversky (1977) by the contrast model. It is assumed that each object is represented by a measurable collection of features and that a function (the matching function), based on the distinctive and common features, exists to quantify the similarities between the pairs of objects. Hierarchical clustering (Sokal and Sneath 1963), additive clustering (Shepard and Arabie, 1979), additive trees (Sattath and Tversky, 1977) and extended trees (Corter and Tversky, 1986) are all special cases of the contrast model.

Some interesting results concerning the problem of choosing the type of model (spatial or nonspatial) to adopt in the applications were obtained in Pruzansky, Tversky and Carroll (1982). It is a common opinion that the analyses based on spatial methods should be accompanied by other techniques based on nonspatial models.

A three-way generalization of the additive clustering model of Shepard and Arabie (1979) was proposed by Carroll and Arabie (1983). The model assumes interobject similarity be an additive function of the common features, which are suitably weighted on each occasion. A common set of clusters (which are allowed to overlap) for all the similarity matrices and the associated sets of weights are obtained by this method.

The estimating procedı ·e combines a mathematical programming technique with ALS and combinatorial optimizations. The INDCLUS program (Carroll and Arabie, 1982) performs the analyses of the method.

Now the case of asymmetric proximities is briefly considered, as in many applications data are observed for which the axiom of symmetry does not hold. Sometimes the problem is removed by averaging the different corresponding data values, particularly when it is possible to suppose the asymmetry to be random error (or noise). However in many situations this characteristic is an intrinsic property of the data that is interesting to analyse. For this reason models and programs of MDS for three-way asymmetric proximities have been introduced. Relevant examples are represented by the DEDICOM (Harshman, 1978) and ASINDSCAL (Young and Lewyckyj, 1979) family of models, based respectively on generalizations of the scalar product and the Euclidean distance models.

Preference data

Preferences are quite common in Psychology, Marketing and other research fields. As a first example we could ask each of a group of people to rank different kinds of products (e.g. liqueurs or newspapers) in order of his/her preference for them. If we collect these data on several occasions (e.g. before, during and after an advertising campaign) we obtain a three-way (three-mode) array. In other applications subjects rank a set of objects with respect to some attributes or characteristics (e.g. sweetness, brightness, noisyness, etc.). As a final example we could observe for each subject both a square matrix of proximities between objects, and ranks of the same objects with respect to one or more attributes.

Examples of analysis of preference data can be found in Shiffman et al. (1981), Carroll (1972), Young and Hamer (1987).

Components Models or Multiway Multidimensional Unfolding Models (De Sarbo 1978, Young and Lewyckyj 1979) can be applied to preferences even if the second model should be preferred for its nonmetric approach which is closer to the nature of the data.

Multidimensional Unfolding (Coombs 1964, Schoneman 1970) is a method for analysing rectangular matrices (i.e. two-way two-mode). By this technique we obtain a representation of the elements of the two modes in such a way that the distances between the elements of different modes are directly interpretable in terms of the entries of the matrix. On the other hand, unlike the Component Models, the distances between elements of the same mode are implicitly "derived" and they do not represent any simple function of the data.

Consider our first example. What we expect from the resulting representation is that each subject-point is located close to points for products the subjects like, and far from points for products the subjects dislike.

Multidimensional Unfolding has been generalized to analyse three-way three-mode matrices in the metric and nonmetric approach (De Sarbo 1978, Young and Lewyckyj 1979, De Sarbo and Carroll 1985). It can be interpreted as a nonsymmetric (nonmetric) generalization of INDSCAL.

An interesting generalization for sorting data using a probabilistic approach is considered by the method CATSCALE described by De Sarbo (this volume).

Another possible approach is External Unfolding. For each subject this method places an "ideal point" in a Euclidean space containing a known object configuration (obtained for instance by applying symmetric MDS to a matrix of similarities among the objects). The ideal points are determined so that their distances from the objects correspond to the expressed preferences. In the third example given above a configuration of the objects-points could be obtained by an MDS method applied to the mean proximity matrix. Then we could carry out an external multiway unfolding analysis considering the observed preference data. In this way a representation of the subjects ("subject space") and a joint plot of objects and attributes are available to explore the data array.

This last approach provides a plot in which the distances between the objects are directly interpretable, moreover it does not suffer from the degeneracies quite common for the other Unfolding methods (Heiser 1981).

Multiway Multidimensional Unfolding models are included in the ALSCAL program. External unfolding is also included, in ALSCAL and in PREFMAP (Carroll 1972, 1980).

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References

Aitkin, M., Francis, B., Raynald, N., Une étude comparative d'analyses des correspondances ou de classifications et de modeles de variables latentes ou de classes latentes, *Revue de Statistique Appliquée,* XXXV, 3 (1987).

Arabie, Ph., J.D. Carroll and W.S. De Sarbo, *Three-way scaling and clustering* (SAGE Publications, Newbury Park, 1987).

Becker, M.P., Clogg, C.C., Analysis of sets of two-way contingency tables using association models, JASA, 84 (1989) 142–151.

Bishop, Y.M., Fienberg, S.E., Holland, P.W., Discrete multivariate analysis: Theory and practice. MIT Press, Cambridge.

Bloxom, B., A note on invariance in three-mode factor analysis, *Psychometrika,* **33** (1968) 347–350.

Bloxom, B., Constrained multidimensional scaling in n-spaces, *Psychometrika,* **43**, (1978) 397–408.

Breiman, L., Friedman, J.H., Estimating optimal transformations for multiple regression and correlation, JASA, 80 (1985) 580–619.

Browne, M.W., *MUTMUM PC user's guide*, Unpublished Report (1990).
Cailliez, F. and J.P. Pages, *Introduction à l'analyse des données* (S.M.A.S.H., Paris, 1976).
Carroll, J.D., Generalization of canonical correlation analysis to three or more sets of variables, *Proc. Am. Psychol. Ass.*, (1968) 227–228.
Carroll, J.D., Individual differences and multidimensional scaling, in: R.N. Shepard et al. (eds.) *Multidimensional scaling: Theory and applications in the behavioral sciences*, Vol. I, (Seminar Press, New York and London, 1972) 105–155.
Carroll, J.D., Models and methods for multidimensional analysis of preferential choice (or other dominance) data in: E.D. Lanterman and H. Feger (eds.) *Similarity and choice* (Hans Huber Publishers, Vienna, 1980) 234–289.
Carroll, J.D. and J.J. Chang, Analysis of individual differences in multidimensional scaling via an *N*-way generalization of "Eckart-Young" decomposition, *Psychometrika*, **35**, (1970) 283–319.
Carroll, J.D. and J.J. Chang, *IDIOSCAL: A generalization of INDSCAL allowing IDIOsyncratic reference systems as well as an analytic approximation to INDSCAL*, Unpublished manuscript (Bell Laboratories, Murray Hill, 1972).
Carroll, J.D. and Ph. Arabie, *How to use INDCLUS, A computer program for fitting the individual differences generalization of the ADCLUS model*, Unpublished manuscript (Bell Laboratories, Murray Hill, 1982).
Carroll, J.D. and Ph. Arabie, INDCLUS: An individual differences generalization of the ADCLUS model and the MAPCLUS algorithm, *Psychometrika*, **48**, (1983) 157–169.
Carroll, J.D., S. Pruzansky and J.B. Kruskal, CANDELINC: A general approach to multidimensional analysis of many-way arrays with linear constraints on parameters, *Psychometrika*, **45** (1980) 3–24.
Cavedon, G., D'Arcangelo, E., De Antoni, F., Analysis of relationships among categorical variables: a comparison between Correspondence Analysis and Log-linear models, *Metron* 40, (1982) 117–144.
Chang, J.J. and J.D. Carroll, *How to use INDSCAL: A computer program for canonical decomposition of N-way cables and individual differences in multidimensional scaling*, Unpublished manuscript (Bell Laboratories, Murray Hill, 1969).
Choulakian, V., Exploratory analysis of contingency tables by log-linear formulation and generalizations of correspondence analysis, *Psychometrika*, 53, (1988) 235–250.
Clogg, C.C., Goodman, L.A., Latent Structure analysis of a set of multidimensional contingency tables JASA, 79 (1984) 762–771.
Commandeur, J.J.F., *Matching configurations*, (DSWO Press, Leiden, 1991).
Coombs, C.H., *A theory of data* (Wiley, New York, 1964).
Coppi, R., Analysis of three-way data matrices based on pairwaise relation measure, in COMPSTAT 1986 – Proc. Computational Statistics, Physica-Verlag, Wien, (1986) 129–139.
Coppi, R., Simultaneous analysis of a set of multi-way contingency tables, in Data Analysis and Informatics (Diday et al. eds.), North-Holland, Amsterdam (1988).
Coppi, R., Di Ciaccio, A., The methodological impact of nonlinear analysis, Rivista di Statistica Applicata, 4, 4 (1993) 407–428.
Coppi, R. and S. Bolasco, *Multiway Data Analysis* (North-Holland, Amsterdam, 1989).
Corter, J.E. and A. Tversky, Extended similarity trees, *Psychometrika*, **51** (1986) 429–451.
Cox, D.R., The analysis of multivariate binary data, *Appl. Statist.*, 21 (1972) 113–120.
D'Ambra, L., Kiers, H., Analysis of log-trilinear models for a three-way contingency table using Parafac/Candecomp, in Atti delle Giornate di Studio Pescara, 11–12 Ottobre 1990 (1991).
D'Ambra, L., Lauro, N., Non symmetrical analysis of three-way contingency tables, in *Multiway Data Analysis*, North-Holland, Amsterdam (1989) 301–316.
D'Ambra, L., Marchetti, G.M., Un metodo per l'analisi interstrutturale di più matrici basato su misure di relazione tra le unità statistiche, Proc. 33-th Scient. Meet. Italian Statistical Society, Cacucci, Bari (1986) 171–182.
Daudin, J.J., Trecourt, P., Analyse factorielle des correspondances et modele Log-lineaire: comparaison de deux methodes sur un example, Rev. Stat. Appl., 28 (1980) 5–24.

De Sarbo, W.S., *Three-way unfolding and situational dependence in consumer preference analysis,* Doctoral Thesis, (University of Pennsylvania, 1978).
De Sarbo, W.S. and J.D. Carroll, Three-way metric unfolding via alternating weighted least squares, *Psychometrika*, **50** (1985).
Di Ciaccio, A., Representation of a new association measure between categories using Multidimensional Scaling, in Data Analysis and Informatics, (E. Diday et al. eds.), North-Holland, Amsterdam (1986).
Di Ciaccio, A., Some considerations on the quantification of categorical data, Research Report 88, Dept. of Data Theory, Leiden (1988).
Escofier, B., Analyse des correspondances multiple conditionelles, in *Data Analysis and Informatics* (E. Diday et al. eds.), North-Holland (1988).
Escofier, B. and J. Pages, *L'Analyse Factorielle Multiple*, Cahiers du BURO n. 42 (Université Pierre et Marie Curie, Paris, 1984).
Escoufier, Y., Le traitement des variables vectorielles, *Biometrics,* **29** (1973) 751–760.
Escoufier, Y., L'analyse conjointe de plusieurs matrices, in: Biométrie et temps 59 (76 édité par la société Francaise de Biométrie, 1980).
Escoufier, Y., Operators related to a data matrix, in: Barra et al. (eds.), *Recent Developments in Statistics*, (North-Holland, Amsterdam, 1977) 125–131.
Escoufier, Y., M.C. Bernard, Ch. Lavit, T. Foucart, A. Barre, B. Fichet, A. Carlier and J.Y. Lafaye, Comparaison d'analyse de tableaux à 3 dimensions à partir d'un exemple, *Statistique et Analyse des Données*, Numéro spécial (1985).
Fraser, C., *COSAN User's Guide*, (The Ontario Institute for Studies in Education, Toronto, 1980).
Friedman, J.H., Multivariate adaptative regression splines, The Annals of Statistics (1991).
Gifi, A., *Nonlinear Multivariate Analysis* (Preliminary edition, Department of Data Theory, Leiden, 1981).
Gifi, A., *Nonlinear Multivariate Analysis* (Wiley, Chicester, 1990).
Gittins, R., *Canonical analysis – A review with applications in ecology* (Springer Verlag, Berlin, 1985).
Glacon, F., *Analyse conjointe de plusieurs matrices de données: comparaison de méthodes*, Thèse de 3ième cycle, Unité de Biométrie, Université de Grenoble (1981).
Goodman, L.A., Exploratory latent structure analysis using both identifiable and unidentifiable models, Biometrika, 61, 2 (1974) 215–231.
Goodman, L.A., Some useful extension of the usual Correspondence analysis approach and the usual log-linear models approach in the analysis of contingency tables, International Statistical Review, 54, 3 (1986) 243–309.
Gower, J.C., Generalized procrustes analysis, *Psychometrika*, **40** (1975) 33–51.
Gower, J.C., Three-way scaling, in. ECAS (ed.) *Methods for Multidimensional Data Analysis*, (Dipartimento di Matematica e Statistica, Università di Napoli, 1987) 273–281.
Gower, J.C., Generalized canonical analysis, in. R. Coppi and S. Bolasco (eds.), *Multiway Data Analysis* (North-Holland, Amsterdam, 1989) 221–232.
Harshman, R.A., Foundations of the PARAFAC procedure: Models and conditions for an "explanatory" multi-modal factor analysis, *UCLA Working Papers in Phonetics 16,* (University Microfilms No. 10,085, 1970) 1–84.
Harshman, R.A., PARAFAC2: Mathematical and technical notes, *UCLA Working Papers in Phonetics 22*, (University Microfilms No. 10,085, 1972) 30–44.
Harshman, R.A., Models for analysis of asymmetrical relationships among N objects or stimuli, Paper presented at the first joint meeting of the Psychometric Society and the Society for Mathematical Psychology, Hamilton, Canada (1978).
Harshman, R.A. and W. De Sarbo, An application of PARAFAC to a small sample problem, demonstrating preprocessing, orthogonality constraints, and split-half diagnostic techniques, in: H.G. Law et al. (eds.) *Research Methods for Multimode Data Analysis* (Praeger, New York, 1984) 602–642.

Heiser, W.J., *Unfolding analysis of proximity data*, Doctoral Dissertation (University of Leiden, 1981).

Horan, C.B., Multidimensional scaling: Combining observations when individuals have different perceptual structures, *Psychometrika,* **34** (1969) 139–165.

Horst, P., Relations among m sets of measures, *Psychometrika,* **26** (1961) 129–149.

Hotelling, H., Relations between two sets of variates, *Biometrika,* **28** (1936) 321–377.

Israels, A.Z., Redundancy Analysis for qualitative data, *Psychometrika,* 49, 3 (1984) 331–346.

Israels, A.Z., *Eigenvalue techniques for qualitative data*, DSWO press Leiden.

Jaffrenou, P.A., *Sur l'analyse des familles finies de variables vectorielles*, Thèse, Université C. Bernard LYON I (1978).

Joreskog, K.G., A general method for the analysis of covariance structures, *Biometrika*, **57** (1970) 239–252.

Joreskog, K.G., Simultaneous factor analysis in several populations, *Psychometrika*, **36**, (1971) 409–426.

Joreskog, K.G. and D. Sorbom, *LISREL 7: A Guide to the program and its applications*, (SPSS Scientific Software Inc., Chicago, 1988).

Kapteyn, A., H. Neudecker and T.J. Wansbeek, An approach to n-mode component analysis, *Psychometrika*, **51** (1986) 269–275.

Keller, W.J., Wansbeek, T., Multivariate methods for quantitative and qualitative data, *Journal of Econometrics*, 22 (1983) 91–111.

Kettenring, J.R., Canonical analysis of several sets of variables, *Psychometrika,* **58** (1971) 433–451.

Kiers, H.A.L., Comparison of "anglo-saxon" and "french" three-mode methods, *Statistique et Analyse des Données*, **13** (1988) 14–32.

Kiers, H.A.L., Three-way methods for the analysis of qualitative and quantitative two-way data, DSWO Press, University of Leiden (1989).

Kiers, H.A.L., *SCA. A program for simultaneous components analysis of variables measured in two or more populations* (IEC Programma, University of Groningen, 1990).

Kiers, H.A.L., Hierarchical relations among three-way methods, *Psychometrika,* **56** (1991) 449–470.

Kiers, H.A.L. and J.M.F. Ten Berge, Alternating least squares algorithms for Simultaneous Components Analysis with equal component weight matrices in two or more populations, *Psychometrika,* **54** (1989) 467–473.

Kroonenberg, P.M., *Three-mode Principal Component Analysis. Theory and Applications* (DSWO Press, Leiden, 1983).

Kroonenberg, P.M., Three-mode component models. A survey of the literature, *Statistica Applicata. Italian Journal of Applied Statistics*, **4** (1992) 619–633.

Kroonenberg, P.M. and J. De Leeuw, Principal component analysis of three-mode data by means of alternating least squares algorithms, *Psychometrika,* **45** (1980) 69–97.

Kruskal, J.B., More factors than subjects, tests, and treatments: an indeterminacy theorem for canonical decomposition and individual differences scaling, *Psychometrika,* **41** (1976) 281–293.

Kruskal, J.B., Three-way arrays: rank and uniqueness of trilinear decompositions, with applications to Arithmetic Complexity and Statistics, *Linear Algebra and Its Applications*, **18** (1977) 95–138.

Kruskal, J.B., Factor Analysis and Principal Components: Bilinear methods, in: W.H. Kruskal and J.M. Tanur (eds.) *International Encyclopedia of Statistics* (Free Press, New York 1978).

Lauro, N., D'Ambra, L., L'analyse non symmetrique des correspondances, in Data Analysis and Informatics, (Diday et al. eds.), North-Holland, Amsterdam, (1983) 433–446.

Law, H.G., C.W. Snyder, J.A. Hattie and R.P. McDonald, *Research methods for multimode data analysis* (Praeger, New York, 1984).

Lazarsfeld, P.F., Henry, N.W., *Latent structure analysis,* Houghton Mifflin, Boston.

Leclerc, A., Chevalier, A., Luce, D., Blanc, M., Analyses des correspondances et modele logistique. possibilities et interet d'approaches complementaires, *Revue de Stat. Appl.*, 33 (1985).

Levin, J., Simultaneous factor analysis of several Gramian matrices, *Psychometrika,* **31** (1966) 413–419.

Lingoes, J.C. and I. Borg, Procustean individual differences scaling: PINDIS, *Journal of Marketing Research*, **13** (1976) 406–407.

Lingoes, J.C. and I. Borg, A direct approach to individual differences scaling using increasingly complex transformations, *Psychometrika,* **43** (1978) 491–519.

Marchetti, G.M., Three-way analysis of two-mode matrices of qualitative data. Research Report, Dept. of Statistics, University of Florence, (1988).

McDonald, R.P., A simple comprehensive model for the analysis of covariance structures, *British Journal of Mathematical and Statistical Psychology*, **31** (1978) 59–72.

McDonald, R.P., The invariant factors model for multimode data, in: H.G. Law et al. (eds.) *Research Methods for Multimode Data Analysis* (Praeger, New York, 1984) 285–307.

Meulman, J., Nonlinear Redundancy Analysis via distances, presented at First Conference of the IFCS, Aachen (1987).

Millsap, R.E. and W. Meredith, Component analysis in cross-sectional and longitudinal data, *Psychometrika,* **53** (1988) 123–134.

Mooijaart, A., Three-factor interaction models by log-trilinear terms in three-way contingency tables, *Statistica Applicata*, 4, 4 (1992) 669–677.

Pruzansky, S., *How to use SINDSCAL: A computer program for individual differences in multidimensional scaling*, Unpublished manuscript, (Bell Laboratories, Murray Hill, 1975).

Pruzansky, S., A. Tversky and J.D. Carroll, Spatial versus tree representations of proximity data, *Psychometrika,* **47** (1982) 3–24.

Ramsay, J.O., Maximum likelihood estimation in multidimensional scaling, *Psychometrika,* **42** (1977) 241–266.

Ramsay, J.O., MULTISCALE, in: S.S. Schiffman et al. (eds.) *Introduction to Multidimensional Scaling: Theory, methods, and applications*, (Academic Press, New York, 1981).

Ramsay, J.O., Some statistical approaches to multidimensional scaling data (with discussion), *Journal Royal Statistical Society (A)*, **145** (1982) 285–312.

Rao, C.R., The use and interpretation of principal component analysis in applied research, *Sankhya A,* **26** (1964) 329–358.

Rizzi, A. and M. Vichi, Relations between sets of variates of a three-way data set, *Statistica Applicata. Italian Journal of Applied Statistics*, **4** (1992) 635–651.

Sabatier, R., Lebreton, J.D., Chessel, D., Principal component analysis with instrumental variables as a tool for modelling composition data, in *Multiway Data Analysis* (Coppi and Bolasco eds.), North-Holland, Amsterdam (1989).

Sands, R., Young, F.W., Component models for three-way data: an alternating least squares algorithm with optimal scaling features. *Psychometrika* 45, (1980) 39–67.

Sattath, S. and A. Tversky, Additive similarity trees, *Psychometrika,* **42** (1977) 319–345.

Schiffman, S.S., M.L. Reynolds and F.W. Young, *Introduction to Multidimensional Scaling: Theory, methods, and applications*, (Academic Press, New York, 1981).

Shepard, R.N. and Ph. Arabie, Additive clustering: representation of similarities as combinations of discrete overlapping properties, *Psychological Review*, **86** (1979) 87–123.

Schonemann, P.H., On metric multidimensional unfolding, *Psychometrika*, **35** (1970) 349–366.

Snyder, Jr. C.W., H.G. Law and J.A. Hattie, Overview of Multimode Analytic Methods, in: H.G. Law et al. (eds.) *Research Methods for Multimode Data Analysis* (Praeger, New York, 1984) 2–35.

Sokal, R.R. and P.H.A. Sneath, *Principles of Numerical Taxonomy* (W.H. Freeman, San Francisco, 1963).

Stewart, D. and W. Love, A general canonical correlation index, *Psychological Bulletin*, **70** (1968) 160–163.

Swaminathan, H., Factor analysis of Longitudinal data, in: H.G. Law et al. (eds.) *Research Methods for Multimode Data Analysis* (Praeger, New York, 1984) 308–332.

Takane, Y., The weighted additive model, in *Research Methods for Multimode Data Analysis,* (Law, H.G. et al. eds), Praeger, New York (1984) 470–517.

Takane, Y., Bozdogan, H., Shibayama, T., Ideal point discriminant analysis, *Psychometrika*, 52 (1987) 371–392.

Ten Berge, J.M.F., Orthogonal procrustes rotation for two or more matrices, *Psychometrika*, **42** (1977) 267–276.

Ter Braak, C.J.F., Canonical correspondence analysis: a new eigenvector technique for multivariate direct gradient analysis. *Ecology,* **67** (1986) 1167–1179.

Tucker, L.R., Implications of factor analysis of three-way matrices for measurement of change, in: C.W. Harris (ed.) *Problems in Measuring Change* (University of Wisconsin Press, Madison, 1963) 122–137.

Tucker, L.R., The extension of factor analysis to three-dimensional matrices, in: N. Frederiksen and H. Gulliksen (eds.) *Contributions to Mathematical Psychology* (Holt, Rinehart and Winston, New York 1964) 110–119.

Tucker, L.R., Some mathematical notes on three-mode factor analysis, *Psychometrika,* **31** (1966) 279–311.

Tucker, L.R., Relations between multidimensional scaling and three-mode factor analysis, *Psychometrika,* **37** (1972) 3–27.

Tucker, L.R. and S. Messick, An individual differences model for multidimensional scaling, *Psychometrika,* **28** (1963) 333–367.

Tversky, A., Features of similarity, *Psychological Review*, **84** (1977) 327–352.

Van der Burg, E., Dijksterhuis, G., Nonlinear canonical correlation analysis of multiway data, in *Multiway Data Analysis* (Coppi and Bolasco eds.) (1989) 245–256.

Van der Burg, E. and G. Dijksterhuis, An application of nonlinear redundancy analysis, *Statistica Applicata. Italian Journal of Applied Statistics*, **4** (1992) 565–575.

Van der Burg, E. and J. De Leeuw, Nonlinear redundancy analysis, *British Journal of Mathematical and Statistical Psychology*, **36** (1990) 217–230.

Van der Burg, E., J. De Leeuw and R. Verdegaal, Nonlinear canonical correlation with m sets of variables, *Psychometrika,* **53** (1988) 171–197.

Van den Wollenberg, A.L., Redundancy analysis an alternative for canonical correlation analysis, *Psychometrika*, **42** (1977) 207–219.

Van de Geer, J.P., Linear relations between k sets of variables, *Psychometrika,* **49**, 79–94.

Van der Heijden, P.M.G., de Leeuw, J., Correspondence analysis used complementary to log-linear analysis, *Psychometrika,* 50, 4, (1985) 429–447.

Van der Heijden, P.M.G., Meijerink, F., Generalized correspondence analysis of multi-way contingency tables and multi-way (super-)indicator matrices, in Multiway Data analysis (Coppi and Bolasco eds.) North-Holland, Amsterdam (1989) 185–202.

Verdegaal, R., *OVERALS,* Research Report UG-86-01, Department of Data Theory, University of Leiden (1986).

Yanay, H., Some generalizations of correspondence analysis in terms of projection operators, in *Data Analysis and Informatics*, (Diday et al. eds.), North-Holland, Amsterdam (1986).

Young, F.W., Quantitative analysis of qualitative data, *Psychometrika,* **46** (1981) 357–388.

Young, F.W., The general euclidean model, in: H.G. Law et al. (eds.) *Research Methods for Multimode Data Analysis* (Praeger, New York, 1984) 440–469.

Young, F.W. and R.M. Hamer, *Multidimensional scaling: Theory and applications* (L.E.A., Hillsdale, 1987).

Young, F.W. and R. Lewyckyj, *ALSCAL User's Guide,* (Data Analysis and Theory Associates, Carrboro, 1979).

Young, F.W., Takane, Y., de Leeuw, J., The principal components of mixed measurement level multivariate data: an alternating least squares method with optimal scaling features, *Psychometrika* **43** (1978) 279–281.