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Three-way fuzzy clustering models for LR fuzzy time trajectories

Renato Coppi, Pierpaolo D'Urso*

Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università degli Studi di Roma "La Sapienza", P.le A. Moro, 5 - 00185 Roma, Italy

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Abstract

Fuzzy multivariate time trajectories are defined. For a suitable class, called LR time trajectories, three types of dissimilarity measures are introduced: the instantaneous, the velocity and the simultaneous measures, respectively. Correspondingly, three different kinds of dynamic fuzzy clustering models are suggested, based on a generalization of the Bezdek and Yang and Ko objective functions for fuzzy clustering. The solutions and characteristics of the three models are then illustrated. A comparative appraisal of their practical meaning is proposed by means of an application to the time pattern of the subjective judgments expressed by a sample of web navigators on different types of banners. Some indications for future research in this methodological domain are finally provided.

Keywords: LR fuzzy time arrays; LR fuzzy time trajectories; Dynamic fuzzy clustering; Cross sectional and/or longitudinal studies; Web-advertising

1. Introduction

Fuzzy clustering was firstly introduced, in a two-way framework, by Bezdek (1974, 1981) and Dunn (1974). Successively, many proposals have been made in this connection (see, e.g., Roubens, 1978; Windham, 1985; Trauwaert, 1987; Hathaway and Bezdek, 1988; Hathaway et al., 1989; Bobrowski and Bezdek, 1991; Jajuga, 1991; Miyamoto and Agusta, 1995; Rousseeuw et al., 1995, 1996; Miyamoto and Umayahara, 1998). In these works, the various authors suggest fuzzy clustering methods in order to classify numerical data (hard or crisp data). Recently, some interesting contributions have

* Corresponding author. Tel.: +39-06-4991-0731; fax: +39-06-406-5808.
E-mail addresses: renato.coppi@uniroma1.it (R. Coppi), pierpaolo.durso@uniroma1.it (P. D'Urso).

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been made concerning interval data, by considering fuzzy (Yang and Ko, 1996; Yang and Liu, 1999; Sato and Sato, 1995), symbolic (Ichino and Yaguchi, 1994; Ravi and Gowda, 1999) or mixed (fuzzy and symbolic) (El-Sonbaty and Ismail, 1998) approaches.

In this paper, we focus our attention on the three-way fuzzy clustering problem, in which the occasions are linked by an ordinal structure. Also in this respect some recent works have been presented (Košmelj, 1986; Košmelj and Batagelj, 1990; Saporta and Lavallard, 1996; Sato and Sato, 1994, 1998; D'Urso, 2001). In particular, here we propose three fuzzy clustering models for a general class of multivariate fuzzy time trajectories: the so-called LR fuzzy time trajectories. Notice that our fuzzy clustering models are three-way extensions of the fuzzy models suggested by Yang and Ko (1996), when the third way is described by time occasions.

Within an "informational approach" (Coppi and D'Urso, 2002), we may state that our models can cope with a twofold typology of fuzziness: fuzzy theoretical information (the method is fuzzy) and fuzzy empirical or observational information (the data are fuzzy).

The structure of the paper is characterized as follows: in Section 2 we define the LR fuzzy time arrays and successively (Sections 3 and 4) the LR fuzzy multivariate time trajectories and the dissimilarity measures allowing the comparison among these trajectories. The fuzzy clustering models are proposed in Section 5. A suggestive application to time data concerning web-advertising is also shown.

2. LR fuzzy time arrays

Following the approach adopted in literature for dealing with fuzzy clustering of (nondynamic) fuzzy data (Yang and Ko, 1996; Yang and Liu, 1999), we distinguish two types of LR fuzzy data time arrays: the LR_1 fuzzy time array (or LR fuzzy numbers time array) and the LR_2 fuzzy time array (or LR fuzzy intervals time array). These are the time three-way extensions of the LR fuzzy data defined, for instance, in Dubois and Prade (1988) and Zimmermann (1996).

Definition 1. An LR_1 fuzzy data time array (same units $\times$ same (fuzzy) variables $\times$ times) is defined as follows:

$$\mathbf{X} \equiv \{x_{ijt} = (c_{ijt}, l_{ijt}, r_{ijt})_{LR}\colon\ i = 1, I; j = 1, J; t = 1, T\},$$

where i, j and t denote the units, variables and times, respectively; $x_{ijt} = (c_{ijt}, l_{ijt}, r_{ijt})_{LR}$ represents the LR_1 fuzzy variable j observed on the ith unit at time t, where c_{ijt} denotes the center and l_{ijt} and r_{ijt} the left and right spread, respectively, with the following membership function:

$$\mu(\tilde{u}_{ijt}) = \begin{cases} L\left(\dfrac{c_{ijt} - \tilde{u}_{ijt}}{l_{ijt}}\right), & \tilde{u}_{ijt} \leqslant c_{ijt}\ (l_{ijt} > 0), \\[2ex] R\left(\dfrac{\tilde{u}_{ijt} - c_{ijt}}{r_{ijt}}\right), & \tilde{u}_{ijt} \geqslant c_{ijt}\ (r_{ijt} > 0), \end{cases}$$

where L (and R) is a decreasing "shape" function from $\Re^+$ to $[0,1]$ with $L(0) = 1; L(z_{ijt}) < 1$ for all $z_{ijt} > 0$, $\forall i,j,t; L(z_{ijt}) > 0$ for all $z_{ijt} < 1$ $\forall i,j,t$; $L(1) = 0$ (or $L(z_{ijt}) > 0$ for all z_{ijt} and $L(+\infty) = 0$).

Moreover, an LR_2 fuzzy data time array is defined as follows:

$$\mathbf{X} \equiv \{x_{ijt} = (c_{1ijt}, c_{2ijt}, l_{ijt}, r_{ijt})_{LR}\colon\ i = 1, I; j = 1, J; t = 1, T\},$$

where $x_{ijt} = (c_{1ijt}, c_{2ijt}, l_{ijt}, r_{ijt})_{LR}$ represents the LR_2 fuzzy variable j observed on the ith unit at time t, c_{1ijt} and c_{2ijt} denote, respectively, the left and right "center" and l_{ijt} and r_{ijt} the left and right spread, respectively, with the following membership function:

$$\mu(\tilde{u}_{ijt}) = \begin{cases} L\left(\dfrac{c_{1ijt} - \tilde{u}_{ijt}}{l_{ijt}}\right), & \tilde{u}_{ijt} \leqslant c_{1ijt}\ (l_{ijt} > 0), \\ 1, & c_{1ijt} \leqslant \tilde{u}_{ijt} \leqslant c_{2ijt}, \\ R\left(\dfrac{\tilde{u}_{ijt} - c_{2ijt}}{r_{ijt}}\right), & \tilde{u}_{ijt} \geqslant c_{2ijt}\ (r_{ijt} > 0). \end{cases}$$

Notice that the concept of membership function belongs to the framework of "fuzzy logic", which is an extension of Boolean logic where the concepts of true and false are replaced by that of partial truth. Boolean logic can be represented by set theory, and in an analogous manner fuzzy logic is represented by fuzzy set theory. This approach was originally developed for the description of natural language (Dubois and Prade, 1988; Everitt et al., 2001; Ruspini et al., 1998; Zadeh, 1965; Zimmermann, 1996).

A particular case of LR_1 fuzzy data time array is the triangular one (with triangular membership function).

In fact, for an LR_1 fuzzy time data array

$$\mathbf{X} \equiv \{x_{ijt} = (c_{ijt}, l_{ijt}, r_{ijt})_{LR}\colon\ i = 1, I; j = 1, J; t = 1, T\},$$

if L and R are of the form

$$L(z) = R(z) = \begin{cases} 1 - z, & 0 \leqslant z \leqslant 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

then $\mathbf{X}$ is a triangular fuzzy time array, with membership function (see Fig. 1):

$$\mu(\tilde{u}_{ijt}) = \begin{cases} 1 - \dfrac{c_{ijt} - \tilde{u}_{ijt}}{l_{ijt}}, & \tilde{u}_{ijt} \leqslant c_{ijt}\ (l_{ijt} > 0), \\ 1 - \dfrac{\tilde{u}_{ijt} - c_{ijt}}{r_{ijt}}, & \tilde{u}_{ijt} \geqslant c_{ijt}\ (r_{ijt} > 0). \end{cases}$$

A particular case of LR_2 fuzzy time data array is the trapezoidal one (with trapezoidal membership function).

In fact, for an LR_2 fuzzy time data array

$$\mathbf{X} \equiv \{x_{ijt} = (c_{1ijt}, c_{2ijt}, l_{ijt}, r_{ijt})_{LR}\colon\ i = 1, I; j = 1, J; t = 1, T\},$$

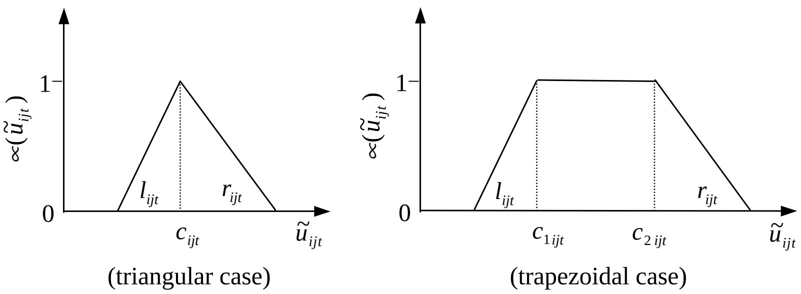

Fig. 1. Triangular and trapezoidal membership functions at time t.

if L and R are of the form

$$L(z)=R(z)=\begin{cases}1-z, & 0\leqslant z\leqslant 1,\\ 0 & \text{otherwise,}\end{cases} \tag{2.2}$$

then $\mathbf{X}$ is a trapezoidal fuzzy time array, with membership function (see Fig. 1):

$$\mu(\tilde{u}_{ijt})=\begin{cases}1-\dfrac{c_{1ijt}-\tilde{u}_{ijt}}{l_{ijt}}, & \tilde{u}_{ijt}\leqslant c_{1ijt}\ (l_{ijt}>0),\\ 1, & c_{1ijt}\leqslant \tilde{u}_{ijt}\leqslant c_{2ijt},\\ 1-\dfrac{\tilde{u}_{ijt}-c_{2ijt}}{r_{ijt}}, & \tilde{u}_{ijt}\geqslant c_{2ijt}\ (r_{ijt}>0).\end{cases}$$

Other particular cases of LR_1 and LR_2 fuzzy time data arrays can be obtained (e.g. with normal or parabolic membership functions) (see Zimmermann, 1996).

For each kind (LR_1 and LR_2) of fuzzy time data array, we can consider the following component-arrays:

LR_1 *case*:

$\mathbf{C}\equiv\{c_{ijt}\colon i=1,I; j=1,J; t=1,T\}$ (time array of the centers);

$\mathbf{L}\equiv\{l_{ijt}\colon i=1,I; j=1,J; t=1,T\}$ (time array of the left spreads);

$\mathbf{R}\equiv\{r_{ijt}\colon i=1,I; j=1,J; t=1,T\}$ (time array of the right spreads).

By combining the indices I,J and T, we can obtain from $\mathbf{X}$ the following stacked fuzzy matrices: $\mathbf{X}\equiv\{\mathbf{X}_i\}_{i=1,I}$, $\mathbf{X}\equiv\{\mathbf{X}_t\}_{t=1,T}$, $\mathbf{X}\equiv\{\mathbf{X}_j\}_{j=1,J}$, with $\mathbf{X}_i\equiv\{x_{ijt}\colon j=1,J; t=1,T\}$, $\mathbf{X}_t\equiv\{x_{ijt}\colon i=1,I; j=1,J\}$, $\mathbf{X}_j\equiv\{x_{ijt}\colon i=1,I; t=1,T\}$.

LR_2 *case*:

$\mathbf{C}_1\equiv\{c_{1ijt}\colon i=1,I; j=1,J; t=1,T\}$ (time array of the left "centers");

$\mathbf{C}_2\equiv\{c_{2ijt}\colon i=1,I; j=1,J; t=1,T\}$ (time array of the right "centers");

$\mathbf{L}$ and $\mathbf{R}$ have the same algebraic form of the time arrays of the left and right spreads defined in the LR_1 case.

3. LR fuzzy multivariate time trajectories

Let R^{J+1} be the vectorial space (space of units), where the axes are referred to the J variables and time. In this space we represent each unit i by means of the following vectors, for each t:

$$_c\mathbf{y}_{it} = (c_{i1t}, \ldots, c_{ijt}, \ldots, c_{iJt}, t)', {}_l\mathbf{y}_{it} = (l_{i1t}, \ldots, l_{ijt}, \ldots, l_{iJt}, t)',$$

$$_r\mathbf{y}_{it} = (r_{i1t}, \ldots, r_{ijt}, \ldots, r_{iJt}, t)' \quad (\mathrm{LR}_1 \text{ case});$$

$$_{c_1}\mathbf{y}_{it} = (c_{1i1t}, \ldots, c_{1ijt}, \ldots, c_{1iJt}, t)', {}_{c_2}\mathbf{y}_{it} = (c_{2i1t}, \ldots, c_{2ijt}, \ldots, c_{2iJt}, t)'$$

$$_l\mathbf{y}_{it} = (l_{i1t}, \ldots, l_{ijt}, \ldots, l_{iJt}, t)', {}_r\mathbf{y}_{it} = (r_{i1t}, \ldots, r_{ijt}, \ldots, r_{iJt}, t)' \quad (\mathrm{LR}_2 \text{ case}).$$

Thus the two following cases can be considered.

1. By fixing t, the scatters

$$_fN_I(t) \equiv \{({}_c\mathbf{y}_{it} \| {}_l\mathbf{y}_{it} \| {}_r\mathbf{y}_{it})\}_{i=1,I} \quad (\mathrm{LR}_1 \text{ case}),$$

$$_fN_I(t) \equiv \{({}_{c_1}\mathbf{y}_{it} \| {}_{c_2}\mathbf{y}_{it} \| {}_l\mathbf{y}_{it} \| {}_r\mathbf{y}_{it})\}_{i=1,I} \quad (\mathrm{LR}_2 \text{ case})$$

   represent the matrix $\mathbf{X}_t$. Letting t vary within its range, the scatters $_fN_I(t)$ are placed on T hyperplanes parallel to the sub-space R^J.
2. By fixing i,

$$_fN_T(i) \equiv \{({}_c\mathbf{y}_{it} \| {}_l\mathbf{y}_{it} \| {}_r\mathbf{y}_{it})\}_{t=1,T} \quad (\mathrm{LR}_1 \text{ case}),$$

$$_fN_T(i) \equiv \{({}_{c_1}\mathbf{y}_{it} \| {}_{c_2}\mathbf{y}_{it} \| {}_l\mathbf{y}_{it} \| {}_r\mathbf{y}_{it})\}_{t=1,T} \quad (\mathrm{LR}_2 \text{ case})$$

   represent the matrix $\mathbf{X}_i$. Each scatter describes the LR_1 (LR_2) fuzzy multivariate time trajectories of object i across the time and

$$\{_fN_T(i) \equiv \{({}_c\mathbf{y}_{it} \| {}_l\mathbf{y}_{it} \| {}_r\mathbf{y}_{it})\}_{t=1,T}\}_{i=1,I} \quad (\mathrm{LR}_1 \text{ case}),$$

$$\{_fN_T(i) \equiv \{({}_{c_1}\mathbf{y}_{it} \| {}_{c_2}\mathbf{y}_{it} \| {}_l\mathbf{y}_{it} \| {}_r\mathbf{y}_{it})\}_{t=1,T}\}_{i=1,I} \quad (\mathrm{LR}_2 \text{ case})$$

   represent, respectively, the set of the LR_1 and LR_2 fuzzy multivariate time trajectories. Each LR_1 (LR_2) fuzzy time trajectory $_fN_T(i)$ crosses the T hyperplanes parallel to R^J.

   A geometrical representation of the triangular version of the previous situations is shown in Coppi and D'Urso (2002).

4. Dissimilarity measures for LR fuzzy multivariate time trajectories

In literature, several geometrical measures and topological aspects have been generalized to the fuzzy framework (Diamond and Kloeden, 1994; Goetshel and Voxman, 1983; Rosenfeld, 1979). By restricting our discussion to suitable distance measures between fuzzy data, we can start by considering the Hausdorff-metric:

$$d_{\mathrm{H}}(A,B)=\max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\sup_{b\in B}\inf_{a\in A}\|a-b\|\right\},$$

where $A,B\subseteq\Re^d$ denote crisp sets (Diamond and Kloeden, 1994; Näther, 2000; Zadeh, 1965; Zimmermann, 1996). By taking into account the so-called α-cuts the Hausdorff-metric d_{H} can be generalized to fuzzy numbers F,G, where $F(\text{or } G):\Re\to[0,1]$:

$$d_q(F,G)=\begin{cases}\left[\int_0^1 (d_{\mathrm{H}}(F_\alpha,G_\alpha))^q\,\mathrm{d}\alpha\right]^{1/q}, & q\in[1,\infty),\\ \sup_{\alpha\in[0,1]} d_{\mathrm{H}}(F_\alpha,G_\alpha), & q=\infty,\end{cases}$$

where the crisp set $F_\alpha\equiv\{x\in\Re^d\colon F(x)\geqslant\alpha\},\alpha\in(0,1]$, is called the α-cut of F (Näther, 2000; Yang and Ko, 1996; Zadeh, 1965).

Another kind of distance measures can be defined via support functions (Näther, 2000).

Then, by considering LR_1 and LR_2 fuzzy data at time t (see Definition 1) different distances can be derived. In particular, we consider the squared Euclidean distances described in the following definition.

Definition 2. With reference to the fuzzy time array **X**, we introduce, for each type of **X**, the following (squared) distances, which are extensions of Yang-Ko's (squared) distance (Yang and Ko, 1996):

LR_1 *case*:

$$ {}_1d^2_{ii't}(\lambda,\rho)=\|\mathbf{c}_{it}-\mathbf{c}_{i't}\|^2+\|(\mathbf{c}_{it}-\lambda\mathbf{l}_{it})-(\mathbf{c}_{i't}-\lambda\mathbf{l}_{i't})\|^2 +\|(\mathbf{c}_{it}+\rho\mathbf{r}_{it})-(\mathbf{c}_{i't}+\rho\mathbf{r}_{i't})\|^2, \tag{4.1}$$

$$\begin{aligned} {}_2d^2_{ii't}(\lambda,\rho)= & \\ & \|(\mathbf{c}_{it}-\mathbf{c}_{it-1})-(\mathbf{c}_{i't}-\mathbf{c}_{i't-1})\|^2\\ & +\|[(\mathbf{c}_{it}-\lambda\mathbf{l}_{it})-(\mathbf{c}_{it-1}-\lambda\mathbf{l}_{it-1})]-[(\mathbf{c}_{i't}-\lambda\mathbf{l}_{i't})-(\mathbf{c}_{i't-1}-\lambda\mathbf{l}_{i't-1})]\|^2\\ & +\|[(\mathbf{c}_{it}+\rho\mathbf{r}_{it})-(\mathbf{c}_{it-1}+\rho\mathbf{r}_{it-1})]-[(\mathbf{c}_{i't}+\rho\mathbf{r}_{i't})-(\mathbf{c}_{i't-1}+\rho\mathbf{r}_{i't-1})]\|^2\\ & =\|{}_c\mathbf{v}_{it}-{}_c\mathbf{v}_{i't}\|^2+\|({}_c\mathbf{v}_{it}-\lambda_l\mathbf{v}_{it})-({}_c\mathbf{v}_{i't}-\lambda_l\mathbf{v}_{i't})\|^2\\ & +\|({}_c\mathbf{v}_{it}+\rho_r\mathbf{v}_{it})-({}_c\mathbf{v}_{i't}+\rho_r\mathbf{v}_{i't})\|^2, \end{aligned} \tag{4.2}$$

where

$$\lambda=\int_0^1 L^{-1}(\omega)\,\mathrm{d}\omega,\quad \rho=\int_0^1 R^{-1}(\omega)\,\mathrm{d}\omega,\quad \mathbf{c}_{it}=(c_{i1t},\dots,c_{ijt},\dots,c_{iJt})',$$

$$\mathbf{c}_{i't}=(c_{i'1t},\dots,c_{i'jt},\dots,c_{i'Jt})',\quad \mathbf{l}_{it}=(l_{i1t},\dots,l_{ijt},\dots,l_{iJt})',$$

$$\mathbf{l}_{i't} = (l_{i'1t}, \ldots, l_{i'jt}, \ldots, l_{i'Jt})', \quad \mathbf{r}_{it} = (r_{i1t}, \ldots, r_{ijt}, \ldots, r_{iJt})',$$

$$\mathbf{r}_{i't} = (r_{i'1t}, \ldots, r_{i'jt}, \ldots, r_{i'Jt})';$$

$$_c\mathbf{v}_{it} = (\mathbf{c}_{it} - \mathbf{c}_{it-1}), \quad _c\mathbf{v}_{i't} = (\mathbf{c}_{i't} - \mathbf{c}_{i't-1}),$$

$$_l\mathbf{v}_{it} = (\mathbf{l}_{it} - \mathbf{l}_{it-1}), \quad _l\mathbf{v}_{i't} = (\mathbf{l}_{i't} - \mathbf{l}_{i't-1}),$$

$$_r\mathbf{v}_{it} = (\mathbf{r}_{it} - \mathbf{r}_{it-1}), \quad _r\mathbf{v}_{i't} = (\mathbf{r}_{i't} - \mathbf{r}_{i't-1})$$

are, respectively, the vectors of the so-called velocities of the centers and left and right spreads pertaining to the fuzzy time trajectory of the ith and i'th units.

LR$_2$ *case*:

$$_1d^2_{ii't}(\lambda,\rho) = \|\mathbf{c}_{1it} - \mathbf{c}_{1i't}\|^2 + \|\mathbf{c}_{2it} - \mathbf{c}_{2i't}\|^2 + \|(\mathbf{c}_{1it} - \lambda\mathbf{l}_{it}) - (\mathbf{c}_{1i't} - \lambda\mathbf{l}_{i't})\|^2 + \|(\mathbf{c}_{2it} + \rho\mathbf{r}_{it}) - (\mathbf{c}_{2i't} + \rho\mathbf{r}_{i't})\|^2, \tag{4.3}$$

$$_2d^2_{ii't}(\lambda,\rho) = \|_{c_1}\mathbf{v}_{it} - {}_{c_1}\mathbf{v}_{i't}\|^2 + \|_{c_2}\mathbf{v}_{it} - {}_{c_2}\mathbf{v}_{i't}\|^2 + \|(_{c_1}\mathbf{v}_{it} - \lambda_l\mathbf{v}_{it}) - (_{c_1}\mathbf{v}_{i't} - \lambda_l\mathbf{v}_{i't})\|^2 + \|(_{c_2}\mathbf{v}_{it} + \rho_r\mathbf{v}_{it}) - (_{c_2}\mathbf{v}_{i't} + \rho_r\mathbf{v}_{i't})\|^2, \tag{4.4}$$

where

$$\mathbf{c}_{1it} = (c_{1i1t}, \ldots, c_{1ijt}, \ldots, c_{1iJt})', \quad \mathbf{c}_{2it} = (c_{2i1t}, \ldots, c_{2ijt}, \ldots, c_{2iJt})',$$

$$\mathbf{c}_{1i't} = (c_{1i'1t}, \ldots, c_{1i'jt}, \ldots, c_{1i'Jt})', \quad \mathbf{c}_{2i't} = (c_{2i'1t}, \ldots, c_{2i'jt}, \ldots, c_{2i'Jt})',$$

$$_{c_1}\mathbf{v}_{it} = (\mathbf{c}_{1it} - \mathbf{c}_{1it-1}), \quad _{c_2}\mathbf{v}_{it} = (\mathbf{c}_{2it} - \mathbf{c}_{2it-1}),$$

$$_{c_1}\mathbf{v}_{i't} = (\mathbf{c}_{1i't} - \mathbf{c}_{1i't-1}), \quad _{c_2}\mathbf{v}_{i't} = (\mathbf{c}_{2i't} - \mathbf{c}_{2i't-1}).$$

Notice that the concept of "velocity" can be defined in the following way. By considering the ith time trajectory of the centers, the velocity, in the time interval $[t-1,t]$, is $_c\mathbf{v}_{it} = (\mathbf{c}_{it} - \mathbf{c}_{it-1})/(t-(t-1)) = (\mathbf{c}_{it} - \mathbf{c}_{it-1})$. Then, for each variable $j, {}_cv_{ijt}$ can be greater (less) than zero according to whether the ith unit presents an increasing (decreasing) rate of change of its position in the time interval $[t-1,t]$; $_cv_{ijt} = 0$ if the unit does not change position from $t-1$ to t. Moreover, note that, for any "component" time trajectory ("center" time trajectory, "lower bound" time trajectory, "upper bound" time trajectory) the velocity pertaining to each pair of successive time points represents the slope of the straight line passing through them: if the velocity is negative (positive) the slope will be negative (positive) and the angle made by each segment of the trajectory with the positive direction of the t-axis will be obtuse (acute) (Coppi and D'Urso, 2000, 2002).

The squared Euclidean distances (4.1) and (4.3) compare, respectively, the positions at time t of the centers (LR$_1$ case) and left and right "centers" (LR$_2$ case) and of the lower and upper bounds (center − left spread and center + right spread) (LR$_1$ and LR$_2$ cases) between each pair of fuzzy time trajectories.

The squared Euclidean distances (4.2) and (4.4) compare, respectively, the slopes (velocities) in each time interval $[t-1,t]$ of the segments of each "component" time trajectory concerning the ith unit with the corresponding slopes of the i'th unit.

Notice that, the previous (squared) distances summarize the fuzziness embodied in each elementary observation of the fuzzy time array **X**, through three parameters (center, left spread, and right spread for the LR_1 case) or four parameters (left and right "centers", left spread, and right spread for the LR_2 case) and the shape of the corresponding membership functions (involving suitable values for the shape-parameters λ and ρ).

On the basis of the above distances we may define appropriate dissimilarity measures between fuzzy multivariate time trajectories, in the following way.

Definition 3. The following dissimilarity measures between $LR_1(LR_2)$ fuzzy multivariate time trajectories are defined:

$$\sum_{t=1}^{T} ({}_1w_{t1} d_{ii't}(\lambda,\rho))^2 \quad \text{(instantaneous dissimilarity measure)}, \tag{4.5}$$

$$\sum_{t=2}^{T} ({}_2w_{t2} d_{ii't}(\lambda,\rho))^2 \quad \text{(velocity dissimilarity measure)}, \tag{4.6}$$

$$\sum_{s=1}^{2} \sum_{t} ({}_sw_{ts} d_{ii't}(\lambda,\rho))^2 \quad \text{(simultaneous dissimilarity measure)}, \tag{4.7}$$

where ${}_1w_t$, ${}_2w_t$, ${}_sw_t$ are suitable weights to be computed in each case (see Section 5).

In particular, dissimilarity (4.5) takes into account, for the LR_1 and LR_2 cases, the (squared) instantaneous distances (4.1) and (4.3), by considering the whole set of the T time occasions. Each occasion is weighted by means of ${}_1w_t$. This weight can be suitably determined in an objective way, as shown in Section 5.

The dissimilarity (4.6) considers, for all time intervals $[t-1,t], t=2,T$, the (squared) velocity distances (4.2) (LR_1 case) and (4.4) (LR_2 case). To each interval a weight ${}_2w_t$ is associated, whose value is computed in an objective manner (see Section 5).

Finally, the dissimilarity measure (4.7) represents, in the observed time domain, a compromise between the (squared) instantaneous and velocity distances. The corresponding weighting system ${}_sw_t$, which is determined within the appropriate clustering procedure (see Section 5), takes simultaneously into account the effects of the single time occasions and time intervals and of the two types of distances (instantaneous and velocity distances).

Concerning the above dissimilarity measures, it is useful to underline the following points:

(1) Attention should be paid to problems of heterogeneity, with particular reference to the variables (different variances and/or units of measurement). An appropriate pre-processing of the data may be required, such as normalization/standardization. In this connection, we can consider different types of pre-processing procedures (Kiers, 2000; Harshman and Lundy, 1984) for centers and (left and right) spreads.

- Centering of the centers, by taking into account the average of the centers. For instance, for the LR_1 case, we can use the following transforms:

$$\tilde{c}_{ijt} = c_{ijt} - \bar{c}_{.jt},$$

$$\tilde{c}_{ijt} = c_{ijt} - \bar{c}_{..t},$$

  where the subscript dot is used to indicate the mean across $i = 1, I(\bar{c}_{.jt})$ and across $i = 1, I$ and $j = 1, J(\bar{c}_{..t})$.
- Normalization of the centers, by dividing the centers, for instance c_{ijt} (LR_1 case), by the normalization factor $\bar{c}_{.j.}$. In this case, we obtain

$$\tilde{\tilde{c}}_{ijt} = \frac{c_{ijt}}{\bar{c}_{.j.}}.$$

- Standardization of the centers, by using, for instance,

$$c^*_{ijt} = \frac{\tilde{c}_{ijt}}{1/I\sqrt{\sum_{i=1}^{I} \tilde{c}^2_{ijt}}}.$$

- Normalization of the spreads, by setting, for example,

$$\tilde{l}_{ijt} = \frac{l_{ijt}}{\bar{l}_{.j.}}, \qquad \tilde{r}_{ijt} = \frac{r_{ijt}}{\bar{r}_{.j.}}.$$

Normalization of the centers and spreads, as illustrated above, is particularly indicated for coping with problems of heterogeneity of units of measurement and/or of size of the variables. In fact, by means of the suggested transform of the data we get a sort of index numbers which allow us to keep the information due to the variability of the different variables (within and across the occasions), while getting rid of possible differences in their overall mean value or in their type of measurement. In any case, when choosing a specific transform of the original data, we should consider the particular informational features we would like to keep in or eliminating from the analysis.

(2) By considering (2.1), (2.2), $\lambda = \int_0^1 L^{-1}(\omega)\,\mathrm{d}\omega, \rho = \int_0^1 R^{-1}(\omega)\,\mathrm{d}\omega$ we obtain, for both the triangular and trapezoidal cases, $\lambda=\rho=\frac{1}{2}$. Then, by substituting $\lambda=\rho=\frac{1}{2}$ in dissimilarities (4.1), (4.2) and (4.3), in the LR_1 or LR_2 cases, we get, respectively, the triangular and trapezoidal versions of the considered dissimilarity measures.

Particular cases of the previous distance and dissimilarity measures (4.1)–(4.7) for fuzzy time trajectories with triangular membership functions have been studied by Coppi and D'Urso (2002). These measures are extensions of the Diamond's distance (Diamond, 1988).

Notice that, other distance measures between fuzzy time trajectories can be obtained by generalizing the measures comparing the respective membership functions. In the static (non dynamic) case, these distances, can be classified according to different approaches (Bloch, 1999; Zwich et al., 1987): the "functional approach", in which the membership functions are compared by means of Minkowski and Canberra distances extended to the fuzzy case (Dubois and Prade, 1983; Kaufman, 1973; Lowen and Peeters, 1998; Pappis and Karacapilidis, 1993); the "information theoretic approach", based on the definition of fuzzy entropy (De Luca and Termini, 1972) and the "set theoretic approach", based on the concepts of fuzzy union and intersection (Chen et al., 1995; Pappis and Karacapilidis, 1993; Wang, 1997; Wang et al., 1995; Zwich et al., 1987).

In the present work, we prefer using the measures (4.1)–(4.7), for the following reasons:

(1) They take into account, simultaneously, the shape of the membership function (through the corresponding parameters λ and ρ) and the component parameters characterizing the examined fuzzy data (centers (LR_1 case), left and right "centers" (LR_2 case), left and right spreads (LR_1 and LR_2 cases)).

(2) A different weighting system is provided for the centers and for the spreads. In particular, as it is reasonable to think, the weights of the centers are larger than those pertaining to the spreads. The stronger contribution of the centers is immediately perceivable when considering, for instance, the symmetric triangular and trapezoidal cases of the squared distances (4.1) and (4.3) (by fixing $\lambda = \rho = \frac{1}{2}$, $\mathbf{l}_{it} = \mathbf{r}_{it}$, $\mathbf{l}_{i't} = \mathbf{r}_{i't}$). In fact, since (4.1) and (4.3) can be rewritten as follows:

$$
{}_1d^2_{ii't}(\lambda,\rho) = \begin{cases} 3(\mathbf{c}_{it}-\mathbf{c}_{i't})'(\mathbf{c}_{it}-\mathbf{c}_{i't}) - 2\lambda(\mathbf{c}_{it}-\mathbf{c}_{i't})'(\mathbf{l}_{it}-\mathbf{l}_{i't}) \\ \quad +\lambda^2(\mathbf{l}_{it}-\mathbf{l}_{i't})'(\mathbf{l}_{it}-\mathbf{l}_{i't}) + 2\rho(\mathbf{c}_{it}-\mathbf{c}_{i't})'(\mathbf{r}_{it}-\mathbf{r}_{i't}) \\ \quad +\rho^2(\mathbf{r}_{it}-\mathbf{r}_{i't})'(\mathbf{r}_{it}-\mathbf{r}_{i't}) & (LR_1 \text{ case}) \\ 2(\mathbf{c}_{1it}-\mathbf{c}_{1i't})'(\mathbf{c}_{1it}-\mathbf{c}_{1i't}) - 2(\mathbf{c}_{2it}-\mathbf{c}_{2i't})'(\mathbf{c}_{2it}-\mathbf{c}_{2i't}) \\ \quad -2\lambda(\mathbf{c}_{1it}-\mathbf{c}_{1i't})'(\mathbf{l}_{it}-\mathbf{l}_{i't}) + \lambda^2(\mathbf{l}_{it}-\mathbf{l}_{i't})'(\mathbf{l}_{it}-\mathbf{l}_{i't}) \\ \quad +2\rho(\mathbf{c}_{2it}-\mathbf{c}_{2i't})'(\mathbf{r}_{it}-\mathbf{r}_{i't}) + \rho^2(\mathbf{r}_{it}-\mathbf{r}_{i't})'(\mathbf{r}_{it}-\mathbf{r}_{i't}) & (LR_2 \text{ case}), \end{cases}
$$

we get, by fixing $\mathbf{s}_{it} \equiv \mathbf{l}_{it} = \mathbf{r}_{it}$ and $\mathbf{s}_{i't} \equiv \mathbf{l}_{i't} = \mathbf{r}_{i't}$:

$$
{}_1d^2_{ii't}\left(\frac{1}{2},\frac{1}{2}\right) = \begin{cases} 3(\mathbf{c}_{it}-\mathbf{c}_{i't})'(\mathbf{c}_{it}-\mathbf{c}_{i't}) + \frac{1}{2}(\mathbf{s}_{it}-\mathbf{s}_{i't})'(\mathbf{s}_{it}-\mathbf{s}_{i't}) \\ \quad (\text{symmetric triangular case}), & (4.8) \\ 2(\mathbf{c}_{1it}-\mathbf{c}_{1i't})'(\mathbf{c}_{1it}-\mathbf{c}_{1i't}) + 2(\mathbf{c}_{2it}-\mathbf{c}_{2i't})'(\mathbf{c}_{2it}-\mathbf{c}_{2i't}) \\ \quad +\frac{1}{2}(\mathbf{s}_{it}-\mathbf{s}_{i't})'(\mathbf{s}_{it}-\mathbf{s}_{i't}) \quad (\text{symmetric trapezoidal case}). & (4.9) \end{cases}
$$

Formulae (4.8) and (4.9) emphasize, immediately, the stronger contribution of the centers. Analogous considerations can be extended to (4.2), (4.4), (4.5), (4.6) and (4.7).

(3) The proposed measures can be applied to a wide family of fuzzy time data (LR fuzzy time arrays).
(4) The (squared) distances (4.1)–(4.4) are well adapted to dynamic situations. In fact, by means of them we may set up appropriate dissimilarity measures (4.5)–(4.7) taking into account the relevant features of the trajectories (namely the location and variation aspects) which could not be captured had we used, instead, other types of distances (e.g., those based on membership functions).
(5) The suggested measures allow an easy implementation of the different dynamic fuzzy clustering algorithms.

In correspondence with the previous dissimilarity measures (4.5), (4.6), (4.7), we propose three different kinds of dynamic fuzzy clustering models:

(1) instantaneous (positional) LR fuzzy clustering model,
(2) velocity (slope) LR fuzzy clustering model,
(3) simultaneous (instantaneous—velocity) LR fuzzy clustering model.

These clustering models can classify a set of fuzzy time trajectories, belonging to the LR family, defining fuzzy partitions that take into account the instantaneous (positional) and/or the velocity characteristics of the trajectories.

5. Dynamic three-way LR fuzzy clustering models

In several real situations, there are many cases in which the data involve subjective or linguistic vagueness (for instance, data based on human perceptions). These data embody a certain degree of fuzziness (fuzzy data). They are nonprecise data or data affected by a source of uncertainty which is not due to randomness (Ruspini et al., 1998). In this section, we analyze the problem of clustering objects for which fuzzy data have been observed over a time period. In addition, we consider a clustering model which is fuzzy too. In the "informational" perspective, we are assuming that both the theoretical information (the model) and the empirical information (the data) are fuzzy. Concerning the fuzzy theoretical information, we observe that "in fuzzy clustering, objects are not assigned to a particular cluster: they possess a membership function indicating the strength of membership in all or some of the clusters" in opposition to the traditional (nonfuzzy) or, in fuzzy clustering jargon, *crisp* or *hard* clustering techniques in which "strength of membership has been either 0 or 1". Moreover, "fuzzy clustering has two main advantages over crisp methods. Firstly, memberships can be combined with other information. In particular, in the special case where memberships are probabilities, results can be combined from different sources using Bayes' theorem. Secondly, the memberships for any given object indicate whether there is a 'second best' cluster that is almost as good as the 'best' cluster, a phenomenon which is often hidden when using other clustering techniques" (Everitt et al., 2001).

Fuzzy clustering for crisp data has been firstly introduced by Bezdek (1974, 1981) and Dunn (1974). Successively, several models have been set up in this connection (e.g., Roubens, 1978; Windham, 1985; Trauwaert, 1987; Hathaway and Bezdek, 1988; Hathaway et al., 1989; Bobrowski and Bezdek, 1991; Jajuga, 1991; Miyamoto and Agusta, 1995; Rousseeuw et al., 1995, 1996; Miyamoto and Umayahara, 1998).

The fuzzy clustering problem for fuzzy data not involving time has been analyzed by different authors (see, for instance, Sato and Sato, 1995; Yang and Ko, 1996). In particular, Yang and Ko (1996) have extended the fuzzy clustering model proposed by Bezdek (1974, 1981), and suggested different fuzzy clustering models for classifying LR fuzzy data.

In particular, the fuzzy clustering problem, in a two-way context, can be formalized in a general form in the following way:

$$\underset{\text{(membership degrees, centroids)}}{\text{minimize}} \sum_{i=1}^{I}\sum_{k=1}^{K} u_{ik}^{m}\tilde{d}_{ik}^{2} \tag{5.1}$$

with the constraints

$$\sum_{k=1}^{K} u_{ik} = 1, \qquad u_{ik} \geqslant 0, \tag{5.2}$$

where u_{ik} denotes the membership degree of the ith unit to the kth cluster and $\tilde{d}_{ik}$ is the Euclidean distance between the ith unit and the centroid which characterizes the kth cluster.

If $\tilde{d}_{ik} =_{\text{crisp}} d_{ik} =$ Euclidean distance between the ith unit and kth centroid computed on crisp data, then model (5.1)–(5.2) is the fuzzy clustering model suggested by Bezdek (1974, 1981).

If $\tilde{d}_{ik} =_{\text{fuzzy}} d_{ik} =$ Euclidean distance between the ith unit and the kth centroid calculated on LR fuzzy data (univariate version of (4.1) and (4.3)), then model (5.1)–(5.2) coincides with one of the possible fuzzy clustering models for (LR_1 and LR_2) fuzzy data proposed by Yang and Ko (1996).

Hence, if $\tilde{d}_{ik} =_{\text{crisp}} d_{ik}$ then the fuzzy clustering model (5.1)–(5.2) defines a fuzzy partition of units based on a crisp data set. If $\tilde{d}_{ik} =_{\text{fuzzy}} d_{ik}$ then the clustering model (5.1)–(5.2) determines a fuzzy partition of units starting from a fuzzy data set.

In this work we propose, in a three-way framework, different dynamic versions of the fuzzy clustering models suggested by Yang and Ko (1996). In particular, the new models enable us to determine a fuzzy partition of fuzzy multivariate time trajectories, by considering suitable time weighting systems, objectively determined by the clustering procedures and computing also the center and the (left and right) spreads of the fuzzy centroid time trajectories.

5.1. Instantaneous LR fuzzy clustering models

We classify a set of LR fuzzy multivariate time trajectories taking into account their instantaneous (positional) features. Distinguishing the LR_1 and LR_2 cases, we have the following cross-sectional fuzzy clustering models for LR_1 and LR_2 multivariate time trajectories with their respective iterative solutions.

5.1.1. LR_1 case

In this case, the fuzzy clustering problem can be formalized in the following way:

$$\underset{\substack{{}_1u_{ik},{}_1w_t,\\ \mathbf{c}_{kt},\mathbf{l}_{kt},\mathbf{r}_{kt}}}{\text{minimize}} \sum_{i=1}^{I}\sum_{k=1}^{K} {}_1u_{ik}^m \sum_{t=1}^{T} ({}_1w_t\,{}_1d_{ikt}(\lambda,\rho))^2$$

$$= \sum_{i=1}^{I}\sum_{k=1}^{K} {}_1u_{ik}^m \sum_{t=1}^{T} \Big({}_1w_t^2(\|\mathbf{c}_{it}-\mathbf{c}_{kt}\|^2 + \|(\mathbf{c}_{it}-\lambda\mathbf{l}_{it})-(\mathbf{c}_{kt}-\lambda\mathbf{l}_{kt})\|^2$$

$$+\|(\mathbf{c}_{it}+\rho\mathbf{r}_{it})-(\mathbf{c}_{kt}+\rho\mathbf{r}_{kt})\|^2) \Big)$$

$$\sum_{k=1}^{K} {}_1u_{ik} = 1, \quad {}_1u_{ik} \geqslant 0; \qquad \sum_{t=1}^{T} {}_1w_t = 1, \quad {}_1w_t \geqslant 0$$

$$\left(\lambda = \int_0^1 L^{-1}(\omega)\,\mathrm{d}\omega, \quad \rho = \int_0^1 R^{-1}(\omega)\,\mathrm{d}\omega\right),$$

where ${}_1u_{ik}$ denotes the membership degree of the ith LR_1 fuzzy multivariate time trajectory with respect to the kth cluster; ${}_1w_t$ is an instantaneous weight; $m > 1$ is a weighting exponent that controls the fuzziness of the obtained fuzzy partition (see Section 6); $\mathbf{c}_{kt}, \mathbf{l}_{kt}, \mathbf{r}_{kt}$ denote, respectively, the vectors of the centers, left and right spreads of the LR_1 fuzzy time trajectory of the kth centroid at time t.

By solving the previous constrained optimization problem (see proof in Appendix A), we get the following iterative solutions:

$${}_1u_{ik} = \frac{1}{\sum_{k'=1}^{K}\left[\dfrac{\sum_{t=1}^{T}({}_1w_t\,{}_1d_{ikt}(\lambda,\rho))^2}{\sum_{t=1}^{T}({}_1w_t\,{}_1d_{ik't}(\lambda,\rho))^2}\right]^{1/(m-1)}},$$

$${}_1w_t = \frac{1}{\sum_{t'=1}^{T}\left[\dfrac{\sum_{i=1}^{I}\sum_{k=1}^{K} {}_1u_{ik}^m\,{}_1d_{ikt}^2(\lambda,\rho)}{\sum_{i=1}^{I}\sum_{k=1}^{K} {}_1u_{ik}^m\,{}_1d_{ikt'}^2(\lambda,\rho)}\right]},$$

$$\mathbf{c}_{kt} = \frac{\sum_{i=1}^{I} {}_1u_{ik}^m[3\mathbf{c}_{it} - \lambda(\mathbf{l}_{it}-\mathbf{l}_{kt}) + \rho(\mathbf{r}_{it}-\mathbf{r}_{kt})]}{3\sum_{i=1}^{I} {}_1u_{ik}^m},$$

$$\mathbf{l}_{kt} = \frac{\sum_{i=1}^{I} {}_1u_{ik}^m(\mathbf{c}_{kt}+\lambda\mathbf{l}_{it}-\mathbf{c}_{it})}{\lambda\sum_{i=1}^{I} {}_1u_{ik}^m},$$

$$\mathbf{r}_{kt} = \frac{\sum_{i=1}^{I} {}_1u_{ik}^m(\mathbf{c}_{it}+\rho\mathbf{r}_{it}-\mathbf{c}_{kt})}{\rho\sum_{i=1}^{I} {}_1u_{ik}^m}.$$

5.1.2. LR_2 case

As in the previous case, the fuzzy clustering model for the LR_2 time trajectories is characterized as follows:

$$\underset{\substack{{}_1u_{ik},{}_1w_t,\\ \mathbf{c}_{1kt},\mathbf{c}_{2kt},\mathbf{l}_{kt},\mathbf{r}_{kt}}}{\text{minimize}} \sum_{i=1}^{I}\sum_{k=1}^{K} {}_1u_{ik}^m \sum_{t=1}^{T} ({}_1w_t\,{}_1d_{ikt}(\lambda,\rho))^2$$

$$=\sum_{i=1}^{I}\sum_{k=1}^{K} {}_1u_{ik}^m \sum_{t=1}^{T} \Big({}_1w_t^2(\|\mathbf{c}_{1it}-\mathbf{c}_{1kt}\|^2 + \|\mathbf{c}_{2it}-\mathbf{c}_{21t}\|^2 + \|(\mathbf{c}_{1it}-\lambda\mathbf{l}_{it})$$

$$-(\mathbf{c}_{1kt}-\lambda\mathbf{l}_{kt})\|^2 + \|(\mathbf{c}_{2it}+\rho\mathbf{r}_{it})-(\mathbf{c}_{2kt}+\rho\mathbf{r}_{kt})\|^2) \Big)$$

$$\sum_{k=1}^{K} {}_1u_{ik}=1, \quad {}_1u_{ik}\geqslant 0; \qquad \sum_{t=1}^{T} {}_1w_t=1, \quad {}_1w_t\geqslant 0$$

$$\left(\lambda=\int_0^1 L^{-1}(\omega)\,\mathrm{d}\omega, \quad \rho=\int_0^1 R^{-1}(\omega)\,\mathrm{d}\omega\right),$$

where $\mathbf{c}_{1kt}, \mathbf{c}_{2kt}$ denote, respectively, the vectors of the left and right "centers" of the LR_2 fuzzy time trajectory of the kth centroid at time t; the other symbols have the same meaning as in the previous model (LR_1 case).

Then, the iterative solutions are (see proof in Appendix B):

$${}_1u_{ik}=\frac{1}{\sum_{k'=1}^{K}\left[\frac{\sum_{t=1}^{T}({}_1w_t\,{}_1d_{ikt}(\lambda,\rho))^2}{\sum_{t=1}^{T}({}_1w_t\,{}_1d_{ik't}(\lambda,\rho))^2}\right]^{1/(m-1)}},$$

$${}_1w_t=\frac{1}{\sum_{t'=1}^{T}\left[\frac{\sum_{i=1}^{I}\sum_{k=1}^{K}{}_1u_{ik}^m\,{}_1d_{ikt}^2(\lambda,\rho)}{\sum_{i=1}^{I}\sum_{k=1}^{K}{}_1u_{ik}^m\,{}_1d_{ikt'}^2(\lambda,\rho)}\right]},$$

$$\mathbf{c}_{1kt}=\frac{\sum_{i=1}^{I}{}_1u_{ik}^m[2\mathbf{c}_{1it}-\lambda(\mathbf{l}_{it}-\mathbf{l}_{kt})]}{2\sum_{i=1}^{I}{}_1u_{ik}^m}, \quad \mathbf{c}_{2kt}=\frac{\sum_{i=1}^{I}{}_1u_{ik}^m[2\mathbf{c}_{2it}+\rho(\mathbf{r}_{it}-\mathbf{r}_{kt})]}{2\sum_{i=1}^{I}{}_1u_{ik}^m},$$

$$\mathbf{l}_{kt}=\frac{\sum_{i=1}^{I}{}_1u_{ik}^m(\mathbf{c}_{ikt}+\lambda\mathbf{l}_{it}-\mathbf{c}_{1it})}{\lambda\sum_{i=1}^{I}{}_1u_{ik}^m}, \quad \mathbf{r}_{kt}=\frac{\sum_{i=1}^{I}{}_1u_{ik}^m(\mathbf{c}_{2it}+\rho\mathbf{r}_{it}-\mathbf{c}_{2kt})}{\rho\sum_{i=1}^{I}{}_1u_{ik}^m}.$$

Notice that the iterative solutions for the LR_1 and LR_2 cases are the same as to the membership degrees and the time weights, very similar as to the left and right spreads, but they differ with respect to the centers.

5.1.3. Iterative algorithms

The iterative algorithm of the clustering model in the LR_1 case is shown below.

Algorithm (LR$_1$ case**).**

Step 1: Fix the parameters $m>1$, K and $\varepsilon>0$ (ε is a small quantity fixed by the researcher), λ, ρ and choose initial membership degrees ${}_1u_{ik}^{(0)}$ and left and right spreads $\mathbf{l}_{kt}^{(0)}$ and $\mathbf{r}_{kt}^{(0)}$ $(i=1,I;\ k=1,K;t=1,T)$.

Step 2: By considering ${}_1u_{ik}^{(0)}$, $\mathbf{l}_{kt}^{(0)}$ and $\mathbf{r}_{kt}^{(0)}$, compute $\mathbf{c}_{kt}^{(0)}$ and ${}_1w_t^{(0)}$.

Step 3: Update ${}_1u_{ik}^{(0)}$ and successively $\mathbf{l}_{kt}^{(0)}, \mathbf{r}_{kt}^{(0)}$ and iterate the recursive procedure.

Step 4: By denoting with ${}_1u_{ik}^{(\upsilon)}$ the membership degree at the υth iteration, compare ${}_1u_{ik}^{(\upsilon)}$ with ${}_1u_{ik}^{(\upsilon+1)}$ using any suitable criterion; for instance, $|{}_1u_{ik}^{(\upsilon+1)}-{}_1u_{ik}^{(\upsilon)}|<\varepsilon$ $(i=1,I;k=1,K)$. If ${}_1u_{ik}^{(\upsilon+1)}$ is sufficiently close to ${}_1u_{ik}^{(\upsilon)}$: stop; otherwise, go back to step 2.

For the LR$_2$ case, we consider $\mathbf{c}_{1kt}, \mathbf{c}_{2kt}$ instead of $\mathbf{c}_{kt}$ and the iterative algorithm follows the same line as in the previous case. These algorithms are analogous to the algorithms utilized for the fuzzy clustering models proposed by Bezdek (1981) (for crisp data) and Yang and Ko (1996) (for fuzzy data) and the same properties hold true. In this connection, we notice that for these models the performances (convergence, etc.) have been suitably investigated by Bezdek (1980), Bezdek et al. (1987) and Yang (1993). Moreover, experimental studies have shown that the fuzzy clustering algorithm of Bezdek is an efficient starting point for the traditional (crisp or hard) clustering procedure (Heiser and Groenen, 1997). Nonetheless, for our clustering models, we performed several "tests" and observed that: the values taken by the objective function in the optimization procedures illustrated in this section decrease monotonically, the iterative algorithms converge quickly to a local minimum after a reasonable number of iterations and present a sensibility to starting points similar to the clustering models suggested by Bezdek (1981) and Yang and Ko (1996) (see also Section 6).

5.2. *Velocity fuzzy clustering models*

In this case the LR fuzzy time trajectories are clustered according to their longitudinal features. In particular, since the longitudinal aspect is represented by the "velocity" of the component time trajectories (center time trajectories and lower and upper bound time trajectories), we consider this particular feature in the clustering procedure. The LR$_1$ and LR$_2$ cases of the velocity clustering models are characterized as follows.

5.2.1. *LR$_1$ case*

In this case, we have the following constrained optimization problem:

$$\underset{\substack{{}_2u_{ik},{}_2w_t,\\ {}_c\mathbf{v}_{kt},{}_l\mathbf{v}_{kt},{}_r\mathbf{v}_{kt}}}{\text{minimize}} \sum_{i=1}^{I}\sum_{k=1}^{K} {}_2u_{ik}^m \sum_{t=2}^{T} ({}_2w_{t2}d_{ikt}(\lambda,\rho))^2$$

$$=\sum_{i=1}^{I}\sum_{k=1}^{K} {}_2u_{ik}^m \sum_{t=2}^{T} \Big({}_2w_t^2(\|{}_c\mathbf{v}_{it}-{}_c\,\mathbf{v}_{kt}\|^2+\|({}_c\mathbf{v}_{it}-\lambda_l\mathbf{v}_{it})-({}_c\mathbf{v}_{kt}-\lambda_l\mathbf{v}_{kt})\|^2$$

$$+\|({}_c\mathbf{v}_{it}-\rho_r\mathbf{v}_{it})-({}_c\mathbf{v}_{kt}+\rho_r\mathbf{v}_{kt})\|^2)\Big)$$

$$\sum_{k=1}^{K} {}_2u_{ik} = 1, \quad {}_2u_{ik} \geqslant 0; \qquad \sum_{t=2}^{T} {}_2w_t = 1, \quad {}_2w_t \geqslant 0$$

$$\left(\lambda = \int_0^1 L^{-1}(\omega)\,\mathrm{d}\omega, \rho = \int_0^1 R^{-1}(\omega)\,\mathrm{d}\omega\right),$$

where w_t is a weight pertaining to time interval $[t-1,t]$; ${}_c\mathbf{v}_{kt}$, ${}_l\mathbf{v}_{kt}$, ${}_r\mathbf{v}_{kt}$ are the vectors of the velocity of the centers, left and right spreads of the LR_1 fuzzy time trajectory of the kth centroid in the time interval $[t-1,t]$.

The iterative solutions are:

$${}_2u_{ik} = \frac{1}{\sum_{k'=1}^{K}\left[\frac{\sum_{t=2}^{T}({}_2w_{t\,2}d_{ikt}(\lambda,\rho))^2}{\sum_{t=2}^{T}({}_2w_{t\,2}d_{ik't}(\lambda,\rho))^2}\right]^{1/(m-1)}},$$

$${}_2w_t = \frac{1}{\sum_{t'=2}^{T}\left[\frac{\sum_{i=1}^{I}\sum_{k=1}^{K}\,{}_2u_{ik\,2}^{m}d_{ikt}^2(\lambda,\rho)}{\sum_{i=1}^{I}\sum_{k=1}^{K}\,{}_2u_{ik\,2}^{m}d_{ikt'}^2(\lambda,\rho)}\right]},$$

$${}_c\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I}\,{}_2u_{ik}^{m}[3\,{}_c\mathbf{v}_{it} - \lambda({}_l\mathbf{v}_{it} - {}_l\mathbf{v}_{kt}) + \rho({}_r\mathbf{v}_{it} - {}_r\mathbf{v}_{kt})]}{3\sum_{i=1}^{I}\,{}_2u_{ik}^{m}},$$

$${}_l\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I}\,{}_2u_{ik}^{m}({}_c\mathbf{v}_{kt} + \lambda\,{}_l\mathbf{v}_{it} - {}_c\mathbf{v}_{it})}{\lambda\sum_{i=1}^{I}\,{}_2u_{ik}^{m}}, \qquad {}_r\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I}\,{}_2u_{ik}^{m}({}_c\mathbf{v}_{it} + \rho_r\mathbf{v}_{it} - {}_c\mathbf{v}_{kt})}{\rho\sum_{i=1}^{I}\,{}_2u_{ik}^{m}}.$$

5.2.2. LR_2 case

The clustering model is

$$\underset{\substack{{}_2u_{ik},{}_2w_t,\\ {}_{c_1}\mathbf{v}_{kt},{}_{c_2}\mathbf{v}_{kt},{}_l\mathbf{v}_{kt},{}_r\mathbf{v}_{kt}}}{\text{minimize}} \sum_{i=1}^{I}\sum_{k=1}^{K}\,{}_2u_{ik}^{m}\sum_{t=2}^{T}({}_2w_{t\,2}d_{ikt}(\lambda,\rho))^2$$

$$= \sum_{i=1}^{I}\sum_{k=1}^{K}\,{}_2u_{ik}^{m}\sum_{t=2}^{T}\big({}_2w_t^2(\|{}_{c_1}\mathbf{v}_{it} - {}_{c_1}\mathbf{v}_{kt}\|^2 + \|{}_{c_2}\mathbf{v}_{it} - {}_{c_2}\mathbf{v}_{kt}\|^2 + \|({}_{c_1}\mathbf{v}_{it} - \lambda_l\mathbf{v}_{it})$$

$$-({}_{c_1}\mathbf{v}_{kt} - \lambda_l\mathbf{v}_{kt})\|^2 + \|({}_{c_2}\mathbf{v}_{it} + \rho_r\mathbf{v}_{it}) - ({}_{c_2}\mathbf{v}_{kt} + \rho_r\mathbf{v}_{kt})\|^2)\big)$$

$$\sum_{k=1}^{K} {}_2u_{ik} = 1, \quad {}_2u_{ik} \geqslant 0; \qquad \sum_{t=2}^{T} {}_2w_t = 1, \quad {}_2w_t \geqslant 0$$

$$\left(\lambda = \int_0^1 L^{-1}(\omega)\,\mathrm{d}\omega, \rho = \int_0^1 R^{-1}(\omega)\,\mathrm{d}\omega\right),$$

where ${}_{c_1}\mathbf{v}_{kt}$, ${}_{c_2}\mathbf{v}_{kt}$ are the vectors of the velocity of the left and right "centers" of the LR_2 fuzzy time trajectory of the kth centroid in the time interval $[t-1,t]$; the other symbols are the same as in the LR_1 case.

In this case the iterative solutions are:

$$
{}_2u_{ik} = \frac{1}{\sum_{k'=1}^{K}\left[\dfrac{\sum_{t=2}^{T}({}_2w_{t2}d_{ikt}(\lambda,\rho))^2}{\sum_{t=2}^{T}({}_2w_{t2}d_{ik't}(\lambda,\rho))^2}\right]^{1/(m-1)}},
$$

$$
{}_2w_t = \frac{1}{\sum_{t'=2}^{T}\left[\dfrac{\sum_{i=1}^{I}\sum_{k=1}^{K}\ {}_2u_{ik}^m\,{}_2d_{ikt}^2(\lambda,\rho)}{\sum_{i=1}^{I}\sum_{k=1}^{K}\ {}_2u_{ik}^m\,{}_2d_{ikt'}^2(\lambda,\rho)}\right]},
$$

$$
{}_{c_1}\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I}\ {}_2u_{ik}^m[2\,{}_{c_1}\mathbf{v}_{it} - \lambda({}_l\mathbf{v}_{it} - {}_l\mathbf{v}_{kt})]}{2\sum_{i=1}^{I}\ {}_2u_{ik}^m}, \qquad {}_{c_2}\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I}\ {}_2u_{ik}^m[2\,{}_{c_2}\mathbf{v}_{it} + \rho({}_r\mathbf{v}_{it} - {}_r\mathbf{v}_{kt})]}{2\sum_{i=1}^{I}\ {}_2u_{ik}^m},
$$

$$
{}_l\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I}\ {}_2u_{ik}^m({}_{c_1}\mathbf{v}_{kt} + \lambda\,{}_l\mathbf{v}_{it} - {}_{c_1}\mathbf{v}_{it})}{\lambda\sum_{i=1}^{I}\ {}_2u_{ik}^m}, \qquad {}_r\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I}\ {}_2u_{ik}^m({}_{c_2}\mathbf{v}_{it} + \rho\,{}_r\mathbf{v}_{it} - {}_{c_2}\mathbf{v}_{kt})}{\rho\sum_{i=1}^{I}\ {}_2u_{ik}^m}.
$$

The iterative algorithms for the velocity fuzzy clustering models for LR_1 and LR_2 fuzzy time trajectories are analogous to those illustrated in the instantaneous models (see Section 5.1). Also the performances are the same.

5.3. Simultaneous fuzzy clustering models

By considering simultaneously the instantaneous (positional) and velocity (slope) aspects of the LR fuzzy time trajectories, we define the following two kinds of clustering models.

5.3.1. LR_1 case

The simultaneous fuzzy clustering model for the LR_1 case is

$$
\text{minimize} \sum_{i=1}^{I}\sum_{k=1}^{K} u_{ik}^m \sum_{s=1}^{2}\sum_{t} ({}_sw_{ts}d_{ikt}(\lambda,\rho))^2
$$

$$
\sum_{k=1}^{K} u_{ik} = 1, \quad u_{ik} \geqslant 0; \qquad \sum_{t} {}_sw_t = 1, \quad {}_sw_t \geqslant 0.
$$

Notice that, in this case, the parameters with respect to which the function has to be minimized are $u_{ik}, {}_sw_t, \mathbf{c}_{kt}, \mathbf{l}_{kt}, \mathbf{r}_{kt}, {}_c\mathbf{v}_{kt}, {}_l\mathbf{v}_{kt}, {}_r\mathbf{v}_{kt}$.

For this model the iterative solutions are:

$$u_{ik} = \frac{1}{\sum_{k'=1}^{K} \left[\frac{\sum_{s=1}^{2} \sum_t ({}_s w_{ts} d_{ikt}(\lambda,\rho))^2}{\sum_{s=1}^{2} \sum_t ({}_s w_{ts} d_{ik't}(\lambda,\rho))^2} \right]^{1/(m-1)}},$$

$${}_s w_t = \frac{1}{\sum_{t'} \left[\frac{\sum_{i=1}^{I} \sum_{k=1}^{K} u_{ik}^m {}_s d_{ikt}^2(\lambda,\rho)}{\sum_{i=1}^{I} \sum_{k=1}^{K} u_{ik}^m {}_s d_{ikt'}^2(\lambda,\rho)} \right]},$$

$$\mathbf{c}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m [3\mathbf{c}_{it} - \lambda(\mathbf{l}_{it} - \mathbf{l}_{kt}) + \rho(\mathbf{r}_{it} - \mathbf{r}_{kt})]}{3 \sum_{i=1}^{I} u_{ik}^m}, \mathbf{l}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m (\mathbf{c}_{kt} + \lambda \mathbf{l}_{it} - \mathbf{c}_{it})}{\lambda \sum_{i=1}^{I} u_{ik}^m},$$

$$\mathbf{r}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m (\mathbf{c}_{it} + \rho \mathbf{r}_{it} - \mathbf{c}_{kt})}{\rho \sum_{i=1}^{I} u_{ik}^m} \qquad (s=1),$$

$${}_c\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m [3 {}_c\mathbf{v}_{it} - \lambda({}_l\mathbf{v}_{it} - {}_l\mathbf{v}_{kt}) + \rho({}_r\mathbf{v}_{it} - {}_r\mathbf{v}_{kt})]}{3 \sum_{i=1}^{I} u_{ik}^m},$$

$${}_l\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m ({}_c\mathbf{v}_{kt} + \lambda {}_l\mathbf{v}_{it} - {}_c\mathbf{v}_{it})}{\lambda \sum_{i=1}^{I} u_{ik}^m}, {}_r\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m ({}_c\mathbf{v}_{it} + \rho {}_r\mathbf{v}_{it} - {}_c\mathbf{v}_{kt})}{\rho \sum_{i=1}^{I} u_{ik}^m} \qquad (s=2).$$

5.3.2. LR_2 case

In this case, we have

$$\text{minimize} \sum_{i=1}^{I} \sum_{k=1}^{K} u_{ik}^m \sum_{s=1}^{2} \sum_t ({}_s w_{ts} d_{ikt}(\lambda,\rho))^2$$

$$\sum_{k=1}^{K} u_{ik} = 1, \quad u_{ik} \geqslant 0, \qquad \sum_t {}_s w_t = 1, \quad {}_s w_t \geqslant 0.$$

Here, the parameters with respect to which the functions have to be minimized are $u_{ik}, {}_s w_t, \mathbf{c}_{1kt}, \mathbf{c}_{2kt}, \mathbf{l}_{kt}, \mathbf{r}_{kt}, {}_{c_1}\mathbf{v}_{kt}, {}_{c_2}\mathbf{v}_{kt}, {}_l\mathbf{v}_{kt}, {}_r\mathbf{v}_{kt}$.

The iterative solutions are:

$$u_{ik} = \frac{1}{\sum_{k'=1}^{K} \left[\frac{\sum_{s=1}^{2} \sum_t ({}_s w_{ts} d_{ikt}(\lambda,\rho))^2}{\sum_{s=1}^{2} \sum_t ({}_s w_{ts} d_{ik't}(\lambda,\rho))^2} \right]^{1/(m-1)}},$$

$${}_s w_t = \frac{1}{\sum_{t'} \left[\frac{\sum_{i=1}^{I} \sum_{k=1}^{K} u_{ik}^m {}_s d_{ikt}^2(\lambda,\rho)}{\sum_{i=1}^{I} \sum_{k=1}^{K} u_{ik}^m {}_s d_{ikt'}^2(\lambda,\rho)} \right]},$$

$$\mathbf{c}_{1kt} = \frac{\sum_{i=1}^{I} u_{ik}^m [2\mathbf{c}_{1it} - \lambda(\mathbf{l}_{it} - \mathbf{l}_{kt})]}{2\sum_{i=1}^{I} u_{ik}^m}, \mathbf{c}_{2kt} = \frac{\sum_{i=1}^{I} u_{ik}^m [2\mathbf{c}_{2it} + \rho(\mathbf{r}_{it} - \mathbf{r}_{kt})]}{2\sum_{i=1}^{I} u_{ik}^m},$$

$$\mathbf{l}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m (\mathbf{c}_{1kt} + \lambda \mathbf{l}_{it} - \mathbf{c}_{1it})}{\lambda \sum_{i=1}^{I} u_{ik}^m}, \mathbf{r}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m (\mathbf{c}_{2it} + \rho \mathbf{r}_{it} - \mathbf{c}_{2kt})}{\rho \sum_{i=1}^{I} u_{ik}^m} \quad (s=1);$$

$$_{c_1}\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m [2_{c_1}\mathbf{v}_{it} - \lambda(_l\mathbf{v}_{it} - _l\mathbf{v}_{kt})]}{2\sum_{i=1}^{I} u_{ik}^m}, _{c_2}\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m [2_{c_2}\mathbf{v}_{it} + \rho(_r\mathbf{v}_{it} - _r\mathbf{v}_{kt})]}{2\sum_{i=1}^{I} u_{ik}^m},$$

$$_l\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m (_{c_1}\mathbf{v}_{kt} + \lambda _l\mathbf{v}_{it} - _{c_1}\mathbf{v}_{it})}{\lambda \sum_{i=1}^{I} u_{ik}^m}, _r\mathbf{v}_{kt} = \frac{\sum_{i=1}^{I} u_{ik}^m (_{c_2}\mathbf{v}_{it} + \rho _r\mathbf{v}_{it} - _{c_2}\mathbf{v}_{kt})}{\rho \sum_{i=1}^{I} u_{ik}^m} \quad (s=2).$$

5.4. Some remarks

We observe that:

- the iterative solutions for the longitudinal and simultaneous LR_1 and LR_2 fuzzy clustering models are obtained analogously to the respective cross-sectional models;
- by setting in the respective cases (LR_1 and LR_2) $\lambda = \rho = \frac{1}{2}$, we obtain the triangular and trapezoidal versions of the different dynamic fuzzy clustering models;
- also for the simultaneous models the iterative algorithms perform similarly to the instantaneous and velocity models.

In conclusion, we make the following points:

(1) For each of the above illustrated clustering models (shown in Sections 5.1, 5.2 and 5.3) a specific choice is made as to the type of membership function and the values of their parameters. The former choice is in fact an assumption, analogous to the model setup in traditional statistical inference. The latter one, concerning the spreads, may either derive from observation (e.g., min and max temperatures registered in a given day) or be made by the researcher on the basis of empirical or theoretical considerations (see also the application in Section 6). It should be noted, in this connection, that the proposed clustering procedures appear to be sufficiently robust with respect to these choices, on the basis of empirical evidence so far collected (systematic simulation studies are planned, in this respect). Finally, it must be remarked that the linear and exponential weights (respectively, the $_sw_t$'s and m), which appear in the objective functions, are optimally determined within the computational procedure.

(2) A comparative assessment of the three types (instantaneous, velocity and simultaneous) of clustering models should be based on the "informational" perspective mentioned in the Introduction. In fact, the instantaneous and the velocity clustering models differ essentially as to the way they deal with the evolutive aspects of the trajectories. The former one looks at the instantaneous distances along

with their time weights, thereby capturing the similarity/dissimilarity between the locations of the trajectories at the various time occasions. The latter model, based on the velocity distances, emphasizes the similarity/dissimilarity between the variations (geometrically, the slopes) observed for each suitably weighted pair of successive times. Obviously, the above aspects constitute two different pieces of information embodied in the data set. When both are considered important the mixed model can be chosen for the clustering task. In this latter model the velocity and instantaneous components are given the same weight. The estimate of their respective parameters is obviously computed from a global optimization point of view. The clusters obtained in this way reflect this choice. However, different weights for the two components might be devised, although this may cause computational difficulties. A final comment refers to the evaluation of the model fit in the above framework. It should be observed that the traditional approach (goodness-of-fit tests, and the like) does not apply in this case. Each clustering model enhances specific informational features in the data. The parameters of the model are then optimally determined (see also Section 6 as to the choice of the number of clusters and the fuzziness coefficient m), and the results interpreted according to the selected informational perspective.

6. An application: web-advertising data

Advertising on Internet is usually done utilizing three different types of banners: "static" banners (which synthesize in a single image text and graphic), "dynamic" banners (characterized by a dynamic gif image, i.e., by a set of images visualized in sequence) and "interactive" banners (which induce the internet-navigators to participate in polls, interactive games and so on).

In view of classifying a set of 18 Web sites ("Iol.it", "Kataweb.it", "Tiscalinet.it", "Msn.it", "Virgilio.it", "Yahoo.it", "Altavista.it", "Excite.it", "Katamail.com", "Altavista.com", "Inwind.it", "Smscash.it", "Ibazar.it", "Repubblica.it", "Mediasetonline.it", "Yahoo.com", "Jumpy.it"), on the basis of the subjective judgments of a sample of 20 Internet navigators concerning the advertising realized by means of different kinds of banners during the time, we have applied the dynamic LR fuzzy clustering models. Note that, for each time, we have considered for the three types of banners of each Web site the median of the judgments expressed by the sample of navigators.

The fuzzy time data array analyzed is

$$\mathbf{X} \equiv \{x_{ijt} = (c_{ijt}, l_{ijt}, r_{ijt}) \colon\ i = 1, \ldots, 18; j = 1, 2, 3; t = 1, \ldots, 6\},$$

where the units are the 18 Web sites and the variables are the subjective judgments on the three kinds (static, dynamic and interactive) of banners observed in six consecutive periods (every fortnight). We have chosen this observation period, because after this period the banners usually lose their effectiveness, i.e., low click through rates are obtained ("banner burnout"). Notice that the sample of Internet navigators is the same for each time (panel data).

Table 1
Linguistic terms and their corresponding triangular fuzzy numbers

Linguistic variable	Fuzzy number
W=Worst	(3, 3, 1)
P=Poor	(4, 1.5, 1.5)
F=Fair	(6, 1, 0.5)
G=Good	(8, 1.75, 0.25)
B=Best	(10, 2, 0)

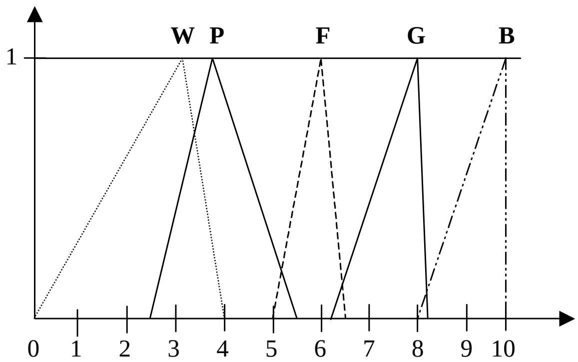

Fig. 2. Triangular fuzzy representation of the linguistic terms.

To take into account the subjective or linguistic vagueness expressed by the human perception a fuzzy coding has been considered (see Table 1 and Fig. 2).

The above fuzzification of the categorical assessments is unavoidably subjective (based on experience or prior observations), although it reflects common sense in interpreting a qualitative scale such as the one considered in Table 1 (see also Liang and Wang, 1991; Liou and Wang, 1994; Raj and Kumar, 1999 for further considerations on this topic). Empirical evidence, so far collected (though further simulation studies are required), supports the assumption of robustness of the fuzzification procedure with respect to the results of the clustering techniques suggested in the present context. For this reason, we have implemented in SAS/IML suitable algorithms, for the different clustering models.

The outputs of our dynamic double fuzzy clustering models are shown in Fig. 3, in which we report: the weighting systems obtained for the three different clustering models, the fuzzy partitions and the graphical representations of the fuzzy partitions. The terminology "double fuzzy" is here utilized with reference to the fuzziness of both the data and the clustering model.

Notice that, according to the previous assumptions, we have considered dynamic triangular fuzzy clustering models ($\lambda=\rho=\frac{1}{2}$). Moreover, in order to determine the number of clusters and the fuzziness coefficient we have suitably extended the cluster-validity

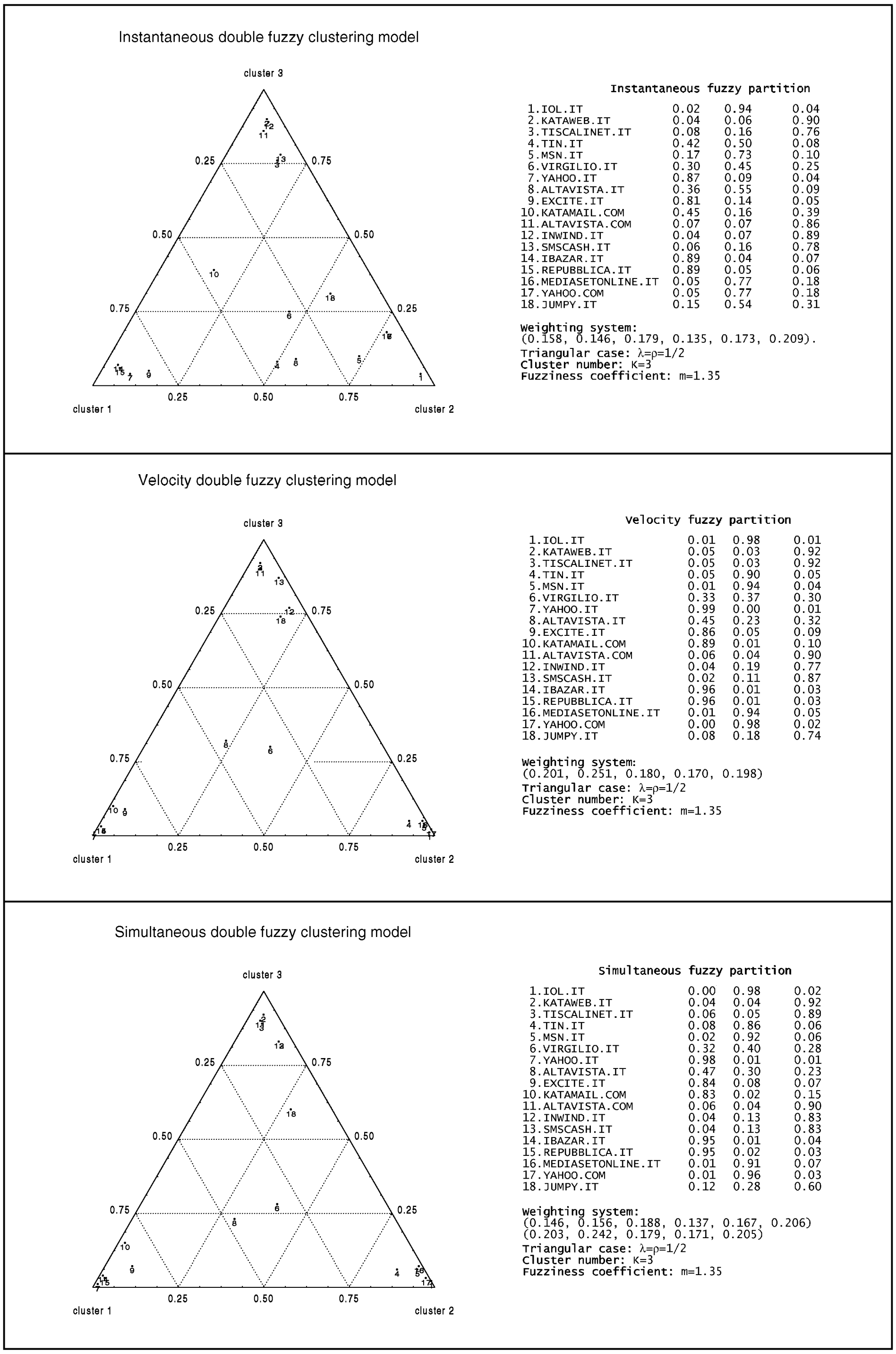

Fig. 3. Fuzzy partitions.

Table 2
Cluster validity for dynamic fuzzy parition

$V_{\lambda,\rho}(m,K)=\dfrac{\sum_{i=1}^{I}\sum_{k=1}^{K}{}_1u_{ik}^{m}\sum_{t=1}^{T}({}_1w_{t1}d_{ikt}(\lambda,\rho))^2}{I\min\limits_{k',k}\sum_{t=1}^{T}({}_1w_{t1}d_{k'kt}(\lambda,\rho))^2},$	$\min\limits_{\substack{m\in M\\ K\in\kappa}} V_{\lambda,\rho}(m,K),$
$\tilde{V}_{\lambda,\rho}(m,K)=\dfrac{\sum_{i=1}^{I}\sum_{k=1}^{K}{}_2u_{ik}^{m}\sum_{t=1}^{T}({}_2w_{t2}d_{ikt}(\lambda,\rho))^2}{I\min\limits_{k',k}\sum_{t=1}^{T}({}_2w_{t2}d_{k'kt}(\lambda,\rho))^2},$	$\min\limits_{\substack{m\in M\\ K\in\kappa}} \tilde{V}_{\lambda,\rho}(m,K),$
$\tilde{\tilde{V}}_{\lambda,\rho}(m,K)=\dfrac{\sum_{i=1}^{I}\sum_{k=1}^{K}u_{ik}^{m}\sum_{s=1}^{2}\sum_{t}({}_sw_{ts}d_{ikt}(\lambda,\rho))^2}{I\min\limits_{k',k}\sum_{s=1}^{2}\sum_{t}({}_sw_{ts}d_{k'kt}(\lambda,\rho))^2},$	$\min\limits_{\substack{m\in M\\ K\in\kappa}} \tilde{\tilde{V}}_{\lambda,\rho}(m,K),$

where

$$
{}_1d^2_{k'kt}(\lambda,\rho)=\begin{cases}\|\mathbf{c}_{kt}-\mathbf{c}_{k't}\|^2+\|(\mathbf{c}_{kt}-\lambda\mathbf{l}_{kt})-(\mathbf{c}_{k't}-\lambda\mathbf{l}_{k't})\|^2+\|(\mathbf{c}_{kt}+\rho\mathbf{r}_{kt})-(\mathbf{c}_{k't}+\rho\mathbf{r}_{k't})\|^2 \quad (LR_1)\\ \|{}_c\mathbf{v}_{kt}-{}_c\mathbf{v}_{k't}\|^2+\|({}_c\mathbf{v}_{kt}-\lambda_l\mathbf{v}_{kt})-({}_c\mathbf{v}_{k't}-\lambda_l\mathbf{v}_{k't})\|^2\\ \quad+\|({}_c\mathbf{v}_{kt}+\rho_r\mathbf{v}_{kt})-({}_c\mathbf{v}_{k't}+\rho_r\mathbf{v}_{k't})\|^2 \quad (LR_2),\end{cases}
$$

$$
{}_1d^2_{k'kt}(\lambda,\rho)=\begin{cases}\|\mathbf{c}_{1kt}-\mathbf{c}_{1k't}\|^2+\|\mathbf{c}_{2kt}-\mathbf{c}_{2k't}\|^2+\|(\mathbf{c}_{1kt}-\lambda\mathbf{l}_{kt})-(\mathbf{c}_{1k't}-\lambda\mathbf{l}_{k't})\|^2\\ \quad+\|(\mathbf{c}_{2kt}+\rho\mathbf{r}_{kt})-(\mathbf{c}_{2k't}+\rho\mathbf{r}_{k't})\|^2 \quad (LR_1)\\ \|{}_{c_1}\mathbf{v}_{kt}-{}_{c_1}\mathbf{v}_{k't}\|^2+\|{}_{c_2}\mathbf{v}_{kt}-{}_{c_2}\mathbf{v}_{kt}\|^2+\|({}_{c_1}\mathbf{v}_{kt}-\lambda_l\mathbf{v}_{kt})-({}_{c_1}\mathbf{v}_{k't}-\lambda_l\mathbf{v}_{k't})\|^2\\ \quad+\|({}_{c_2}\mathbf{v}_{kt}+\rho_r\mathbf{v}_{kt})-({}_{c_2}\mathbf{v}_{k't}+\rho_r\mathbf{v}_{k't})\|^2 \quad (LR_2),\end{cases}
$$

$M=$ set of possible values of m,
$\kappa=$ set of possible values of K.
Note: Obviously, for the triangular case we consider only the LR_1 case, with $\lambda=\rho=\frac{1}{2}$.

criterion proposed by Xie and Beni (1991) (see Table 2), obtaining for each of the three cases the same number of clusters ($K=3$) and fuzziness coefficient ($m=1.35$).

Concerning the results (see Fig. 3), we note that, for the three different kinds of dynamic fuzzy clustering, very similar fuzzy partitions have been obtained. In fact, many Internet Sites present trajectories with similar instantaneous locations and/or velocities.

In particular, by considering the instantaneous fuzzy partition, we notice that the following Web-sites belong to the first class (characterized by interactive banners with medium-good judgment during time) with a high membership degree: Yahoo.it (0.87), Excite.it (0.81), Ibazar.it (0.89), Repubblica.it (0.89). In the second cluster (mainly represented by Web-sites with dynamic banner showing an "alternating" behavior) we record: Iol.it (0.94), Msn.it (0.73), Mediasetonline.it (0.77) and Yahoo.com (0.77). Finally, in the third cluster (characterized, chiefly, by Web-sites with static banner with medium-good judgment over the time) the highest membership degrees are obtained for Kataweb.it (0.90), Tiscalinet.it (0.76), Altavista.com (0.86), Inwind.it (0.89) and Smscash.it (0.78).

The previous results are substantially confirmed in the velocity fuzzy partition, though with a smaller degree of fuzziness.

We notice that the simultaneous fuzzy partition summarizes properly the results obtained by taking into account the instantaneous (positional) and velocity (shape) features of the examined time trajectories. Moreover, in the simultaneous case the ordinal structure of the weighting system is essentially equivalent to those obtained in the previous two models. This is essentially due to the above-mentioned similarity between the location and variation aspects of the observed trajectories. Finally, by considering the Web-advertising data set, we tested the computational performances of the suggested fuzzy clustering models, fixing $m = 1.35$, $K = 3$, $\lambda = \rho = \frac{1}{2}$ and $\varepsilon = 0.0001$. In order to perform the algorithm, the starting points have been generated randomly by a uniform distribution and the following stopping rules have been adopted: $|{}_1u_{ik}^{(\upsilon+1)} -{}_1 u_{ik}^{(\upsilon)}| < \varepsilon$ (instantaneous model), $|{}_2u_{ik}^{(\upsilon+1)} -{}_2 u_{ik}^{(\upsilon)}| < \varepsilon$ (velocity model) and $|u_{ik}^{(\upsilon+1)} - u_{ik}^{(\upsilon)}| < \varepsilon$ (simultaneous model). In Fig. 4, for each of the three models, we show different graphics, in which we represent, respectively:

- the variations of the number of iterations needed to reach a local minimum solution for different values of the weighting exponent m (that controls the fuzziness of the fuzzy partition);
- the value of the objective function for different values of m;
- the value of the objective function for different starting points (for $m = 1.35$) (we have tried 20 starting points have been tried and a continuous representation has been utilized for an easier visualization);
- the value of the objective function for different iteration cycles (for $m = 1.35$) (in order to show that the objective function decreases monotonically).

The results shown in Fig. 4 confirm substantially the computational characteristics indicated in Section 5. In particular, for each of the considered models, the value of the objective function decreases monotonically with increasing m and with increasing number of cycles (for $m = 1.35$). Instead, it remains constant over different starting points (for $m = 1.35$). Moreover, we note that the number of iterations, for all models, is reasonably small.

7. Concluding remarks

A highly flexible approach to clustering multivariate time trajectories has been described. The main features of this approach are as follows:

(1) The explicit recognition of the common informational nature of the ingredients of the data analytic procedure (and of the uncertainty associated with them, here treated according to a fuzzy perspective): the data and the clustering model.
(2) The adoption of a suitable class of membership functions representing the fuzziness of the observed trajectories (i.e., the LR functions).

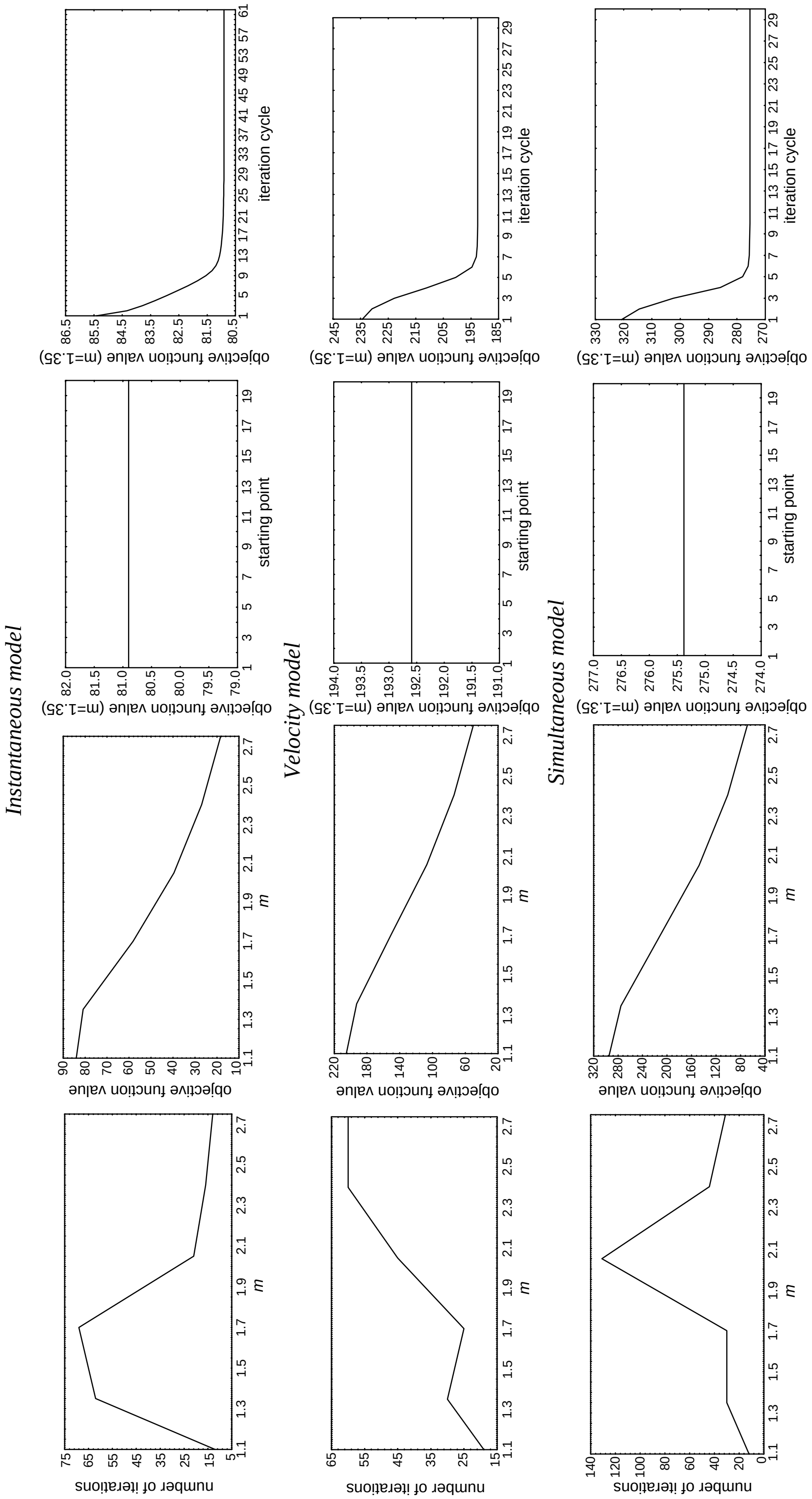

Fig. 4. Computational performances of the suggested dynamic fuzzy clustering models with the web-advertising data set.

(3) The construction of appropriate dissimilarity measures between fuzzy trajectories, taking into account various features such as instantaneous location and dynamic evolution.
(4) The extensive use of a generalized Bezdek criterion as the basis for the clustering process.
(5) The possibility of applying the proposed clustering models in various observational settings, including the case where qualitative data are collected (such as subjective judgments, ordinal categories, mixed data). Obviously, this requires an adequate fuzzification of the qualitative data, as illustrated in Section 6.

One or more of the above mentioned features can be suitably modified in view of improving the performance of the proposed class of clustering models (e.g., the dissimilarity measures or the clustering criterion). This is the subject of future work in this field of methodological research.

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Appendix A. Proof of the iterative solutions for the cross-sectional LR_1 fuzzy clustering model

Let us fix the values of $\mathbf{c}_{kt}, \mathbf{l}_{kt}, \mathbf{r}_{kt}$ and ${}_1w_t$. By considering the Lagrangian function

$$L(m,\lambda,\rho)=\sum_{i=1}^{I}\sum_{k=1}^{K}{}_1u_{ik}^m\sum_{t=1}^{T}({}_1w_{t\,1}d_{ikt}(\lambda,\rho))^2-\delta\left(\sum_{k=1}^{K}{}_1u_{ik}-1\right)$$

(where δ is the Lagrange multiplier) and setting the first derivatives with respect to ${}_1u_{ik}$ and δ equal to zero, we get, respectively, $m\,{}_1u_{i'k'}^{m-1}\sum_{t=1}^{T}({}_1w_{t\,1}d_{i'k't}(\lambda,\rho))^2-\delta=0$ and $\sum_{k=1}^{K}{}_1u_{ik}-1=0$ and then we obtain ${}_1u_{ik}$.

Analogously, let us fix ${}_1u_{ik}$ and $\mathbf{c}_{kt}, \mathbf{l}_{kt}, \mathbf{r}_{kt}$. By considering the Lagrangian function

$$\tilde{L}(m,\lambda,\rho)=\sum_{i=1}^{I}\sum_{k=1}^{K}{}_1u_{ik}^m\sum_{t=1}^{T}({}_1w_t d_{ikt}(\lambda,\rho))^2-\gamma\left(\sum_{t=1}^{T}{}_1w_t-1\right)$$

(where γ is the Lagrange multiplier) and setting the first derivatives with respect to ${}_1w_t$ and γ equal to zero we have: $2\,{}_1w_{t'}\sum_{i=1}^{I}\sum_{k=1}^{K}{}_1u_{ik}^m\,{}_1d_{ikt'}^2(\lambda,\rho)-\gamma=0$ and $\sum_{t=1}^{T}{}_1w_t-1=0$ and then ${}_1w_t$ can be derived.

Finally, in order to obtain $\mathbf{c}_{kt}$, $\mathbf{l}_{kt}$ and $\mathbf{r}_{kt}$, we solve the following unconstrained optimization (minimization) problem with respect to $\mathbf{c}_{kt}$, $\mathbf{l}_{kt}$ and $\mathbf{r}_{kt}$:

$$\underset{\mathbf{c}_{kt},\mathbf{l}_{kt},\mathbf{r}_{kt}}{\text{minimize}}\sum_{i=1}^{I}\sum_{k=1}^{K}u_{ik}^m\sum_{t=1}^{T}({}_1w_{t\,1}d_{ikt}(\lambda,\rho))^2$$

$$
= \sum_{k=1}^{K} \sum_{t=1}^{T} {}_1w_t^2 \left[\underset{\mathbf{c}_{kt},\mathbf{l}_{kt},\mathbf{r}_{kt}}{\text{minimize}} \sum_{i=1}^{I} u_{ik}^m \, {}_1d_{ikt}^2(\lambda, \rho) \right]
$$

$$
= \sum_{k=1}^{K} \sum_{t=1}^{T} {}_1w_t^2 \left[\underset{\mathbf{c}_{kt},\mathbf{l}_{kt},\mathbf{r}_{kt}}{\text{minimize}} F_{kt}(m, \lambda, \rho) \right], \tag{A.1}
$$

where

$$
F_{kt}(m, \lambda, \rho) = \sum_{i=1}^{I} u_{ik}^m \, {}_1d_{ikt}^2(\lambda, \rho),
$$

$$
\begin{aligned}
{}_1d_{ikt}^2(\lambda, \rho) &= \|\mathbf{c}_{it} - \mathbf{c}_{kt}\|^2 + \|(\mathbf{c}_{it} - \lambda \mathbf{l}_{it}) - (\mathbf{c}_{kt} - \lambda \mathbf{l}_{kt})\|^2 + \|(\mathbf{c}_{it} + \rho \mathbf{r}_{it}) - (\mathbf{c}_{kt} + \rho \mathbf{r}_{kt})\|^2 \\
&= 3(\mathbf{c}_{it} - \mathbf{c}_{kt})'(\mathbf{c}_{it} - \mathbf{c}_{kt}) - 2\lambda(\mathbf{c}_{it} - \mathbf{c}_{kt})'(\mathbf{l}_{it} - \mathbf{l}_{kt}) + \lambda^2(\mathbf{l}_{it} - \mathbf{l}_{kt})'(\mathbf{l}_{it} - \mathbf{l}_{kt}) \\
&\quad + 2\rho(\mathbf{c}_{it} - \mathbf{c}_{kt})'(\mathbf{r}_{it} - \mathbf{r}_{kt}) + \rho^2(\mathbf{r}_{it} - \mathbf{r}_{kt})'(\mathbf{r}_{it} - \mathbf{r}_{kt}).
\end{aligned}
$$

Then, setting the first derivatives of $F_{kt}(m, \lambda, \rho)$ with respect to $\mathbf{c}_{kt}, \mathbf{l}_{kt}, \mathbf{r}_{kt}$ equal to zero, we get the iterative solutions for $\mathbf{c}_{kt}, \mathbf{l}_{kt}$ and $\mathbf{r}_{kt}$.

Appendix B. Proof of the iterative solutions for the cross-sectional LR$_2$ fuzzy clustering model

The membership degrees ${}_1u_{ik}$ and the time weights ${}_1w_t$ are obtained analogously to LR$_1$. Also $\mathbf{c}_{1kt}$, $\mathbf{c}_{2kt}$, $\mathbf{l}_{kt}$ and $\mathbf{r}_{kt}$ are obtained as in the LR$_1$ case, by solving the minimization problem (A.1), in which

$$
\begin{aligned}
{}_1d_{ikt}^2(\lambda, \rho) &= \|\mathbf{c}_{1it} - \mathbf{c}_{1kt}\|^2 + \|\mathbf{c}_{2it} - \mathbf{c}_{21t}\|^2 + \|(\mathbf{c}_{1it} - \lambda \mathbf{l}_{it}) - (\mathbf{c}_{1kt} - \lambda \mathbf{l}_{kt})\|^2 \\
&\quad + \|(\mathbf{c}_{2it} + \rho \mathbf{r}_{it}) - (\mathbf{c}_{2kt} + \rho \mathbf{r}_{kt})\|^2 \\
&= 2(\mathbf{c}_{1it} - \mathbf{c}_{1kt})'(\mathbf{c}_{1it} - \mathbf{c}_{1kt}) + 2(\mathbf{c}_{2it} - \mathbf{c}_{21t})'(\mathbf{c}_{2it} - \mathbf{c}_{21t}) \\
&\quad - 2\lambda(\mathbf{c}_{1it} - \mathbf{c}_{1kt})'(\mathbf{l}_{it} - \mathbf{l}_{kt}) + \lambda^2(\mathbf{l}_{it} - \mathbf{l}_{kt})'(\mathbf{l}_{it} - \mathbf{l}_{kt}) \\
&\quad + 2\rho(\mathbf{c}_{2it} - \mathbf{c}_{21t})'(\mathbf{r}_{it} - \mathbf{r}_{kt}) + \rho^2(\mathbf{r}_{it} - \mathbf{r}_{kt})'(\mathbf{r}_{it} - \mathbf{r}_{kt}).
\end{aligned}
$$

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