PSYCHOMETRIKA—VOL. 47, NO. 1.
MARCH, 1982

A MULTIDIMENSIONAL SCALING MODEL FOR THE SIZE-WEIGHT ILLUSION

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The kinds of individual differences in perceptions permitted by the weighted euclidean model for multidimensional scaling (e.g., INDSCAL) are much more restricted than those allowed by Tucker's Three-mode Multidimensional Scaling (TMMDS) model or Carroll's Idiosyncratic Scaling (IDIOSCAL) model. Although, in some situations the more general models would seem desirable, investigators have been reluctant to use them because they are subject to transformational indeterminacies which complicate interpretation. In this article, we show how these indeterminacies can be removed by constructing specific models of the phenomenon under investigation. As an example of this approach, a model of the size-weight illusion is developed and applied to data from two experiments, with highly meaningful results. The same data are also analyzed using INDSCAL. Of the two solutions, only the one obtained by using the size-weight model allows examination of individual differences in the strength of the illusion; INDSCAL can not represent such differences. In this sample, however, individual differences in illusion strength turn out to be minor. Hence the INDSCAL solution, while less informative than the size-weight solution, is nonetheless easily interpretable.

Key words: individual differences, multidimensional scaling, three-mode factor, INDSCAL, size-weight illusions.

Introduction

Models for three-way MDS

Several models have been proposed for studying individual differences in multidimensional scaling. The first such model was the Tucker and Messick [1963] "points of view" approach, based on an Eckart and Young [1936] resolution of the N by $n(n-1)/2$ matrix of interpoint distances. This model has been superceded by more general models which have overcome weaknesses pointed out by Ross [1966].

The weighted euclidean model. Horan [1969] proposed an individual differences model for multidimensional scaling in which the subjects gave different weights to the axes of a common stimulus space. Thus if d_{jki} is the psychological distance for person i between stimulus j and stimulus k, then the model can be written as

$$d_{jki}^2 = \sum_{t=1}^{r} w_{it}^2 (b_{jt} - b_{kt})^2 \tag{1}$$

where b_{jt} is the projection of stimulus j on dimension t, and w_{it} is a weight indicating the importance person i gives to dimension t. This representation has come to be known as the weighted euclidean model. Horan [1969] developed a procedure for estimating the configuration of points (i.e., a set of b_{jt} values) given dissimilarity matrices from several subjects,

This paper is based on the first author's doctoral dissertation at the Department of Psychology, University of Illinois at Urbana-Champaign. The aid of Professor Ledyard R Tucker is gratefully acknowledged.

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0033-3123/82/0300 – 2666$00.75/0

but his procedure did not estimate the subject weights. Carroll and Chang [1970] developed an iterative procedure called INDSCAL (for Individual Differences Scaling) which would estimate both the common stimulus projections and the subject weights. Bloxom [1968] also developed an estimation procedure for this model. Procedures which fit the weighted euclidean model have an important special property; they provide a unique solution under quite general conditions, when some minor indeterminacies are arbitrarily removed. [For further discussion and/or proofs of this uniqueness property see (Harshman, 1970, 1972; Harshman & Berenbaum, 1981; Kruskal, 1976, 1977; Harshman, Note 1.)]

The weighted euclidean model has proved very popular, as indicated by at least fifty applications in the literature. This popularity would seem to be due, in part, to the fact that there are no "rotational" indeterminacies to resolve, and also to the fact that its solutions have proved interpretable in a wide range of multidimensional scaling applications. Recently, new methods of fitting the model to data have been developed, including Ramsey's [1977] maximum likelihood procedure and Takane, Young, and de Leeuw's [1977] nonmetric procedure. Thus, there is every reason to expect that the use of this model will continue to increase.

More general models. Several models have appeared in the literature which allow for individual differences more complicated than the simple differential weighting of dimensions permitted by the weighted euclidean model. Carroll and Chang [Note 2] and Carroll and Wish [1974] have presented a model called IDIOSCAL. Tucker [1972] has developed a model called three mode multidimensional scaling (TMMDS) as a special case of three mode factor analysis in which two of the modes are identical. Harshman [1972] has proposed a model called PARAFAC2, as a generalization of his three-way factor analysis model PARAFAC [Harshman, 1970].

The IDIOSCAL, TMMDS and PARAFAC2 models can all be considered special cases of the following very general expression:

$$\mathbf{X}_i = \mathbf{B}\mathbf{H}_i\mathbf{B}' \tag{2}$$

where $\mathbf{X}_i$ is the ith subject's scalar products matrix (normally derived from a matrix of judged dissimilarities by adding a constant, then squaring all entries and double-centering the matrix), $\mathbf{B}$ represents a common stimulus space, and $\mathbf{H}_i$, named the individual characteristic matrix by Tucker, relates the common stimulus space to the scalar products.

The three models differ in the way in which the $\mathbf{H}_i$ matrices are constrained. In all three models, the $\mathbf{H}_i$ are symmetric positive definite or semidefinite matrices. The IDIOSCAL model places no additional restrictions on the parameters in $\mathbf{H}_i$, but in the TMMDS and PARFAC2 representations the $\mathbf{H}_i$ become increasingly constrained. In TMMDS, the $\mathbf{H}_i$ matrices are computed from estimates of the core matrix and person space of a three mode factor analysis [Tucker, 1972]. That is

$$h_{pp'i} = \sum_{q=1}^{s} g_{pp'q} z_{qi} \tag{3}$$

where $g_{pp'q}$ is taken from the core matrix and z_{qi} is from the person space matrix. In essence, this relation requires that each subject's $\mathbf{H}_i$ be some linear combination of a set of matrices $\mathbf{G}_q$, which are slices of the core matrix. Consequently, the variations in $\mathbf{H}_i$ are more constrained in TMMDS than in IDIOSCAL unless there are a large number of $\mathbf{G}_q$, i.e., unless s, the dimensionality of the person space, is greater than or equal to $r(r + 1)/2$, where r is the dimensionality of the stimulus space. Finally, the PARAFAC2 model places the most severe restrictions on H_i. It requires that all $\mathbf{H}_i = \mathbf{D}_i\mathbf{H}\mathbf{D}_i$, where $\mathbf{H}$ is a matrix of dimensional interrelationships common to all subjects and $\mathbf{D}_i$ is a diagonal weighting matrix specific to the ith subject. (For discussion of further models that can be considered part of this same series, see Harshman, Note 1).

IDIOSCAL, TMMDS and PARAFAC2 also differ in the way in which the $\mathbf{H}_i$ matrices are interpreted. In the IDIOSCAL model, the $\mathbf{H}_i$ matrices are interpreted in terms of an orthogonal rotation of the common stimulus space, followed by a weighting of the rotated dimensions. That is

$$\mathbf{H}_i = \mathbf{T}_i\mathbf{W}_i^2\mathbf{T}_i' \tag{4}$$

where $\mathbf{T}_i$ is an orthogonal matrix and $\mathbf{W}_i$ is diagonal.

In the TMMDS and PARFAC2 models, the $\mathbf{H}_i$ matrices are interpreted as weighted cosine matrices, that is

$$\mathbf{H}_i = \mathbf{W}_i\mathbf{R}_i\mathbf{W}_i \tag{5}$$

where $\mathbf{W}_i$ is a diagonal weight matrix and $\mathbf{R}_i$ is a matrix of cosines showing the ith subject's perceived relations among the dimensions of the weighted object space. In TMMDS the $\mathbf{R}_i$ can differ across subjects, but in PARAFAC2 a common matrix of cosines among dimensions is assumed to hold for all subjects.

The Problems of Indeterminacy and Interpretation

The greater generality of the IDIOSCAL and TMMDS models entails a substantial indeterminacy in the form of the solution. This indeterminacy complicates the job of interpreting the results of an IDIOSCAL or TMMDS analysis, since it means that part of the interpretation will presumably involve transformation of the solution to a "preferred" form. This problem is analogous to the rotation problem in factor analysis, but in two respects is even more acute in three-way MDS: (a) the principles (such as simple structure) which are used to guide selection of a preferred solution in factor analysis are often less clearly applicable to multidimensional scaling studies; and (b) the transformational possibilities for IDIOSCAL and TMMDS are more numerous and more complex than in traditional two-mode factor analysis. Any nonsingular transformation of the stimulus space $\mathbf{B}$ is permissible, provided the compensatory inverse transformations are applied to the $\mathbf{H}_i$. One can also consider transformations of the person characteristic space (defined by the set of $\mathbf{W}_i$); indeed, such a transformation will be used in the size-weight analysis presented later in this article. Finally, one might seek transformations of the TMMDS solution which result in a core matrix that has some desired form, e.g., approximately "diagonal" (Cohen, Note 3; McCallum, 1976b).

On the other hand, the lack of indeterminacy of the INDSCAL model can cause consternation when it appears that a much more interpretable solution might be obtained by a rotation of the solution. Rotation is only possible within the INDSCAL model for those dimensions whose weights exhibit what Carroll and Wish [1974] call a parallel pattern, implying a reduced rank in the weight matrix. Obviously, some deviations from perfect parallelism might be attributed to random error and the solution still subjected to permissible rotations, but just how much deviation is acceptable, and in what other conditions, if any, it might be justifable to rotate an INDSCAL solution, are issues which deserve further study and discussion.

PARAFAC2 is not subject to the same degree of indeterminacy as IDIOSCAL and TMMDS. In general, transformations or rotations of the common stimulus space will not preserve the proportionality relationships among the $\mathbf{H}_i$ matrices specified by the PARAFAC2 model. It has been conjectured that PARAFAC2 possesses the same kind of uniqueness as INDSCAL [Harshman, 1972 but cf. Carroll & Wish, 1974] but proving this has turned out to be more difficult than expected, as has the development of a "well behaved" algorithm for fitting the model to data. (However, some recent theoretical results appear encouraging, see Harshman, Note 1). In any case, it is clear that PARAFAC2 has much less indeterminacy than IDIOSCAL and TMMDS; it is also a less general model, not permitting the range of individual variations allowed by IDIOSCAL and TMMDS. But the size-weight model depends on transformation of an indeterminate general solution to a special form. Consequently, PARAFAC2 is unsuitable for the development of such a model and will not be considered further in this article.

We have seen that those general purpose models which provide the richest description of individual differences in perceptions of the stimulus space (IDIOSCAL and TMMDS), do so at the expense of introducing indeterminacy problems which have not been adequately solved. The present study is aimed, in part, at presenting a method of dealing with such problems, in cases where one has some *a priori* theories concerning the perceptual processes involved. By considering the types of individual differences anticipated in the specific domain under study, it is sometimes possible to construct "special purpose" models of intermediate generality between INDSCAL and IDIOSCAL or TMMDS. Such models can be tailored to represent the psychological processes under investigation, and can test for patterns of individual variation more general than the weighted euclidean model would permit, while avoiding many of the problems associated with indeterminacies of the unrestricted IDIOSCAL or TMMDS models. As an example of this process, we shall develop a "special purpose" three way multidimensional scaling model of the size-weight illusion.

Theoretical Development of the Size-Weight Model

Lifted Weights Experiments and the Size-Weight Illusion

The size-weight illusion is the well known phenomenon in which small objects feel heavier than large objects having the same mass. (In the discussion that follows, the term "heaviness" will be used to refer to the subject's perception of an object's downward pull, under the influence of the size-weight illusion; "weight" will be used to refer to the subject's perception of an object's downward pull, in the absence of the size-weight illusion e.g., when size and weight are not perceived simultaneously.) Since this effect of size on heaviness is likely to vary from one subject to the next, it seems likely that an adequate multidimensional scaling representation of the size-weight illusion will require patterns of individual differences in perception more complicated than those permitted by the weighted euclidean model. At the same time, the phenomenon may be simple enough to permit a "special purpose" model to be constructed to deal with these patterns of inter-subject variation. Consequently, the size-weight illusion would appear to provide an appropriate domain for demonstrating our proposed approach to overcoming the indeterminacies of the very general three-way scaling models. We shall begin by briefly reviewing the previously published one-mode and two -mode studies of the illusion before proceeding to develop our proposed three-mode model.

Unidimensional studies. While the size-weight illusion has been known for some time, it is only recently that quantitative models of the illusion have been experimentally tested. As a result of these tests some controversy has arisen as to whether an additive or a ratio model of the illusion is more appropriate. Sjoberg [1969], and Stevens and Ruben [1970] used magnitude estimation procedures and found that their data gave a good fit to a ratio or density model of the illusion. Ross and DiLollo [1968] proposed a "vector model" of psychophysical judgments and demonstrated that the model gave a good account of the data from three magnitude estimation experiments. Anderson [1970] used a category rating task and found that these data were more closely fitted by an additive model of the illusion. Birnbaum and Veit [1974] obtained category ratings of differences in heaviness. They found inconsistencies between both the ratio and additive models and their data. Feeling that the additive model was more plausible, they speculated that contextual effects might have produced the deviations from the additive model and proposed a study to test for such contextual effects. Sarris and Heineken [1976] found that the additive model was supported by category rating data and the ratio model was supported by magnitude estimation data. Thus in the area of lifted weights experiments, as in other areas of psychophysical experiment, two approaches to obtaining response scales (category ratings and magnitude estimation) give rise to contrary results. Because the additive model of the illusion is more congenial to the multidimensional scaling approach which we wish to adopt, we will emphasize this perspective and use the category scaling method of obtaining data for our tests.

Multidimensional studies. Since the psychological processes involved in the illusion depend on both size and weight, some investigators have replaced the unidimensional analysis of subjective heaviness with multidimensional scaling techniques, in an attempt to understand better the relationship between size and weight in the production of the illusion.

Donovan and Ross [1969] asked subjects to make magnitude estimations of differences in heaviness. They carried out a multidimensional scaling analysis on the average matrix of difference estimates and found two (or more) dimensions. Although the results weren't completely clear, one dimension was apparently heaviness and the other dimension may have been density.

Harshman (Note 4) considered the case where subjects made "global dissimilarity" judgments for pairs of lifted weights taking into account both subjective size and subjective heaviness. He pointed out that such a task could generate data which violated the weighted euclidean model, and that such violations could take at least two forms: (a) the composition of the perceptual dimensions of size and heaviness could vary from subject to subject (as the strength of the illusion varied); and (b) when assessing the dissimilarity between two stimuli, a subject might not utilize his subjective size and subjective heaviness dimensions independently, but instead a conceptual association between size and heaviness might result in oblique perceptual axes. To look for these effects, he conducted a lifted weights experiment in which subjects rated differences between stimuli varying in size and mass. He also had the same subjects do unidimensional scaling of size, heaviness, and weight (i.e., subjective heaviness when the stimuli were concealed behind a curtain). In all four tasks, category rating scales were used. Two-way multidimensional scaling was then applied separately to each subject's data to recover his or her stimulus configuration. By regressing the two-dimensional MDS solution for each subject upon his or her independent scales of size, weight, and heaviness, evidence for both types of violations of the weighted euclidean model was uncovered. There were moderate variations in the influence of the illusion on subjective heaviness and there were oblique relations among those directions in the MDS space which corresponded to size, heaviness, and weight. These oblique relations appeared to vary across subjects, and split-half analysis indicated that these individual differences were at least moderately reliable.

Modelling the Size-Weight Illusion

Let us now develop a "special purpose" three-way multidimensional scaling model for size-weight illusion data. Assume that subjects are asked to give "global" dissimilarity ratings of pairs of stimuli varying in size and mass. The axes of the common stimulus space for such data should be some function of subjective size and subjective weight (in the absence of the illusion). If we assume an additive model of the illusion, then the transformation of the size-weight space to a size-heaviness space will be linear and hence the dimensionality of the space after the illusion exerts its effect will be unchanged. For simplicity, let us assume that the stimuli are constructed in a three by three factorial design, similar to that used by Birnbaum and Veit [1974] in their Experiment 1. The theoretical stimulus space, in the absence of the size-weight illusion, will thus be a rectangle. If the levels of size and mass are chosen so as to be approximately equally spaced on a subjective scale, then the theoretical stimulus space will be approximately square. [This square stimulus space will resemble the theoretical stimulus space used by MacCallum (1976a) in his study of the effect of oblique-axis data on the INDSCAL solution.] This stimulus space is shown in Figure 1. In the present context, dimension 1 corresponds to the subjective size of the stimuli and dimension 2 corresponds to the subjective weight (i.e., in the absence of the size-weight illusion). If the three levels of size are called small (S), medium (M) and big (B) and the three levels of mass are called light (L), medium (M) and heavy (H), then the stimuli can be coded SL for small, light; BM for big, medium, etc. To develop a simple model of each individual's perception of this theoretical stimulus space, we will first assume that there is no weight-size illusion. That is, we assume that the mass of a stimulus does not affect its perceived size. Then the additive size-weight illusion can be represented as a rotation of dimension 2 in Figure 1 to 2′. Projections of the stimuli onto dimension 2′ give a subjective heaviness dimension in the presence of the size-weight illusion. The angle of rotation θ_i could be different for each subject and would be a measure of the extent to which the subject experienced the illusion.

This transformation of the theoretical stimulus projections, given by the columns of **B**, into stimulus projections $\mathbf{B}_i$ which include the effect of the size-weight illusion, is effected by a transformation matrix $\mathbf{T}_i$ for subject i. That is

$$\mathbf{B}_i = \mathbf{B}\mathbf{T}_i. \tag{6}$$

Each $\mathbf{T}_i$ matrix is of the form

$$\mathbf{T}_i = \begin{bmatrix} 1 & -\sin\theta_i \\ 0 & \cos\theta_i \end{bmatrix}. \tag{7}$$

In making his or her judgments, each subject may give differential weights to the size and

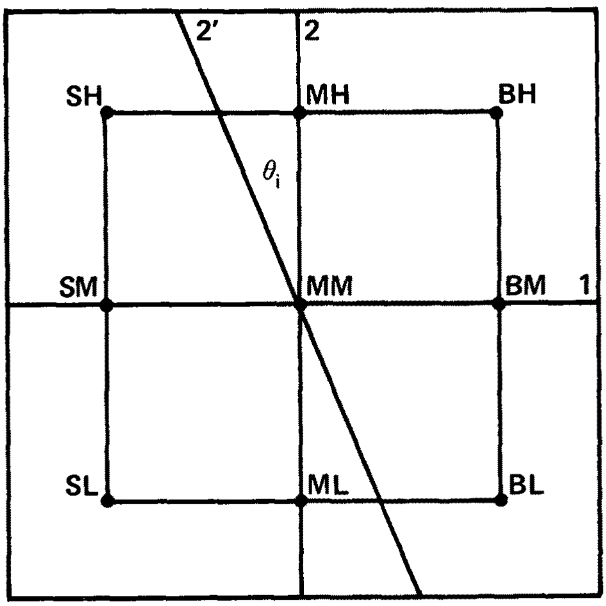

FIGURE 1
Theoretical Stimulus Space: Dimension 1—Size; Dimension 2—Weight (without illusion).

heaviness of the stimuli. This differential weighting can be represented by a diagonal matrix $\mathbf{W}_i$ of weights for subject i. Then if $\mathbf{F}_i$ gives the perceptual dimensions for subject i,

$$\mathbf{F}_i = \mathbf{B}_i\mathbf{W}_i \tag{8}$$

and from (6) and (8)

$$\mathbf{F}_i = \mathbf{B}\mathbf{T}_i\mathbf{W}_i. \tag{9}$$

Now if we ignore for simplicity the possibility of oblique use of these dimensions, we can plot the two columns of $\mathbf{F}_i$ as orthogonal axes in a perceptual space. Figure 2 shows a hypothetical subject's perceptual space $\mathbf{F}_i$ under the model. Dimension 1 is a perceptual size dimension, and all stimuli having the same size have equal projections on the dimension. Dimension 2 is a perceptual heaviness dimension; because of the illusion, stimuli of different size, but the same mass, have different projections on the dimension, with smaller stimuli having higher projections.

If we assume that subjects use the dimensions of the perceptual space independently (i.e., orthogonally) then we can write an expression for X_i, the scalar product matrix for subject i, as follows:

$$\mathbf{X}_i = \mathbf{F}_i\mathbf{F}_i' \tag{10}$$

(ignoring error terms for the moment). From (9) and (10) we can obtain

$$\mathbf{X}_i = \mathbf{B}\mathbf{T}_i\mathbf{W}_i^2\mathbf{T}_i'\mathbf{B}'. \tag{11}$$

Let

$$\mathbf{H}_i = \mathbf{T}_i\mathbf{W}_i^2\mathbf{T}_i \tag{12}$$

then from (11) and (12)

$$\mathbf{X}_i = \mathbf{B}\mathbf{H}_i\mathbf{B}'. \tag{13}$$

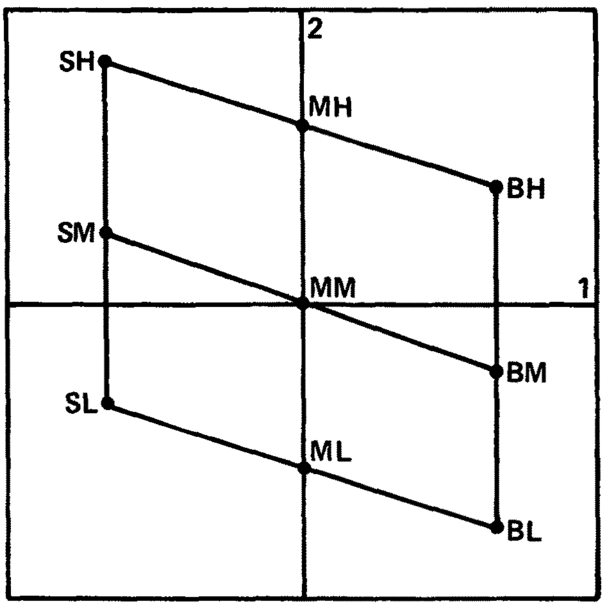

FIGURE 2
Theoretical Perceptual Space: Dimension 1—Size; Dimension 2—Heaviness (with illusion).

Equations (6) through (13) develop a model for the size-weight illusion in which (12) gives the decomposition of the person characteristic matrices. While this resembles the IDIOSCAL decomposition, it differs in that the transformation is not orthogonal. While the orthogonal transformation matrix of the IDIOSCAL model would be difficult to interpret in the context of the size-weight illusion, the oblique transformation matrix proposed in the present model is highly interpretable. In addition, the common stimulus space is determined and is not arbitrary. Of course, the model is specific to the size-weight illusion and cannot be applied in general as the IDIOSCAL and TMMDS models can. Hence, this model will be referred to in the discussion that follows as the size-weight illusion (SWI) model. It is a representation of the perceptual process by which the perceived size and weight of an object interact to produce a perception of the size and heaviness of the object. The only requirement that the data must meet in order to fit the model is that the $\mathbf{H}_i$ matrices must be positive definite. Then an upper triangular matrix $\mathbf{U}_i$ can be found such that

$$\mathbf{H}_i = \mathbf{U}_i \mathbf{U}_i'. \tag{14}$$

Let

$$\mathbf{W}_i^2 = diag(\mathbf{U}_i \mathbf{U}_i'), \tag{15}$$

then

$$\mathbf{T}_i = \mathbf{U}_i \mathbf{W}_i^{-1}, \tag{16}$$

so that

$$\mathbf{U}_i = \mathbf{T}_i \mathbf{W}_i. \tag{17}$$

Substituting (17) into (14) produces (12).

An Empirical Application of the Model

Data Collection

Data appropriate for the SWI model were obtained by having subjects rate the dissimilarity of pairs of stimuli varying in size and weight.

Subjects. Two experiments were conducted using the same subjects in each. There were nineteen subjects (ten females, nine males). One was a graduate student, and the others were fulfilling an introductory psychology course requirement.

Stimuli. There were nine stimuli arranged in a three by three, size by mass factorial design. The stimuli were plastic blocks, painted flat black with metal lifting rings on top. The blocks were approximately cubic with sides 50, 65, and 88 millimeters. The three levels of mass were 100, 150, and 225 grams.

Procedure. A similar procedure was used in both experiments. Each subject was tested individually with about one week between each experimental session. *S* sat at a table and was presented with a pair of stimuli. *S* picked up the stimuli one in each hand, keeping his elbows on the table. *S* then responded with a rating from a rating scale. Each session lasted for about one hour during which a short warm-up series of trials was presented, followed by two replications of the full set of stimuli. *E* shuffled cards specifying the stimuli pairs to randomize the order of presentation. *S* responded orally and *E* wrote the responses on the card.

Experiment 1. In experiment one, subjects made dissimilarity ratings on a rating scale ranging from zero (exactly alike), through one (very similar) to nine (very dissimilar). *S*s were asked to pay equal attention to size and heaviness in making their ratings. There were 72 ratings per replicate, as identical pairs of stimuli were not presented.

Experiment 2. The rating scale was the same as that used in Experiment 1. *S*s were allocated at random to two instructional conditions. Group *H* was instructed to make dissimilarity ratings paying more attention to heaviness, whereas Group *S* was instructed to pay more attention to size. As in experiment 1, there were 72 ratings per replicate.

Analyses and Results—Experiments 1 and 2

SWI model analysis. The data from Experiments 1 and 2 were analyzed both separately and together. The combined analysis produced results that were nearly identical to the individual analyses, so only the combined analysis results will be presented here.

Each subject's four ratings for each pair of stimuli were averaged to produce a nine by nine symmetric dissimilarity matrix with zeroes in the diagonal. (The averaging would presumably remove any order effects induced by subjects being biased toward one hand.) The stimulus space and person characteristic matrices were obtained using Tucker's [1972] TMMDS procedure. Preliminary TMMDS analyses indicated that the stimulus space was three rather than two dimensional, and that additive constants were required to convert the dissimilarites to distances. Additive constants for three dimensions were computed separately using Cooper's [1972] procedure. Prior to input to the TMMDS procedure the scalar products matrix for each subject was scaled so that the entries had a variance of one. The nonzero eigenvalues for the stimulus space were 2151.62, 595.32, 178.60, 47.05, 30.70, 27.71, 23.86, and 23.15. These eigenvalues decline more slowly after the first three, indicating a three dimensional stimulus space which accounted for 95.05 percent of the stimulus space variance. The first ten eigenvalues for the subject spaces were 2554.96, 287.16, 46.31, 35.86, 23.28, 17.96, 16.34, 14.53, 13.80, and 9.75. These eigenvalues drop off gradually after the first two, indicating a two dimensional subject space which accounted for 92.34 percent of the subject space variance. Thus the core matrix was three by three by two.

The next question to be dealt with was that of transformations. The third dimension in the stimulus space was found to be a second size dimension. The "size space" was two dimensional, with the points arranged in the shape of a triangle, rather than the unidimensional "size space" anticipated by the model of the size-weight illusion developed above. The second size dimension apparently arose because of nonlinearity in the contributions of levels of size to the dissimilarity ratings. The reason for this nonlinearity is not clear; however, as this dimension was considered to be less important than the first two dimensions, it was decided to find transformations that would minimize its influence on the modelling of the size-weight illusion.

The first transformation carried out on the stimulus space was an oblique procrustean transformation. The target matrix comprised normalized contrast vectors for the three levels of size, the three levels of mass plus a vector contrasting the medium size with the small and large. After the procrustes transformation, the third dimension was further rotated in the plane of the first and third dimensions to minimize the sum of squares of off-diagonal entries for the third stimulus dimension in the core matrix. The final transformed stimulus space is shown in Table 1 and is plotted in Figure 3. The first two dimensions of the stimulus space seem to fit the theoretical space quite well, with the fit for the size dimension being particularly good.

We next considered the person space. This space is described by a rectangular matrix with a row for each person and a column for each person-dimension or idealized person [see Tucker, 1972]. Nonsingular transformations of the person space, followed by inverse transformations of the core matrix, leave the $\mathbf{H}_i$ person characteristic matrices unchanged. However, such transformations can be very useful in interpreting the subjects' behavior.

TABLE 1

Final Transformed Stimulus Space
SWI Analysis - Experiments 1 & 2

Stimulus	Dimension 1	2	3
1 SL	-.4097	-.4297	.1222
2 SM	-.3952	.0227	.0847
3 SH	-.4191	.4025	.0748
4 ML	-.0085	-.3782	-.4192
5 MM	.0012	-.0136	-.4395
6 MH	.0063	.4011	-.4764
7 BL	.4114	-.3719	.3295
8 BM	.4080	-.0839	.3051
9 BH	.4057	.4510	.4187

Examination of the subject space showed that Group *H* and Group *S* subjects from Experiment 2 form two distinct clusters, with the Experiment 1 points in between. The subject space was rotated by putting axes through the centroids of the Group *H* and Group *S* points. Figure 4 is a plot of the transformed person space. Group *H* and *S* points are represented by the letters *H* and *S* respectively, and the Experiment 1 points are represented by 1's.

The core matrix was transformed by the inverses of the transformations applied to the stimulus and person spaces. The transformed core matrix is shown in Table 2. The core matrix can be interpreted in conjunction with the person space. Idealized person 1 corresponds to the Group *H* subjects and idealized person 2 corresponds to the Group *S* subjects.

For both idealized persons, the off-diagonal entries for stimulus dimension 3 are quite small, indicating that this dimension was perceived as orthogonal to the first two stimulus dimensions. Thus in modelling the $\mathbf{H}_i$ matrices for stimulus dimension 3, it was only necessary to obtain a subject weight for the dimension.

For idealized person 1, the plane of the core matrix corresponding to the first two stimulus dimensions shows large diagonal entries and a reasonably large negative off-diagonal entry. A negative off-diagonal entry is implied by the model of the size-weight illusion presented above, since an increase in stimulus size reduces stimulus heaviness. For idealized person 2, the diagonal entry for stimulus dimension 1 is larger and the diagonal entry for dimension 2 is smaller than for the first idealized person. Since the person space was transformed so that the first idealized person corresponds to the cluster of Group *H* subjects, (i.e., those who were instructed to pay more attention to heaviness) while the second idealized person corresponds to the cluster of Group *S*, subjects (i.e., those instructed to pay more attention to size) we can see that the instructional conditions apparently

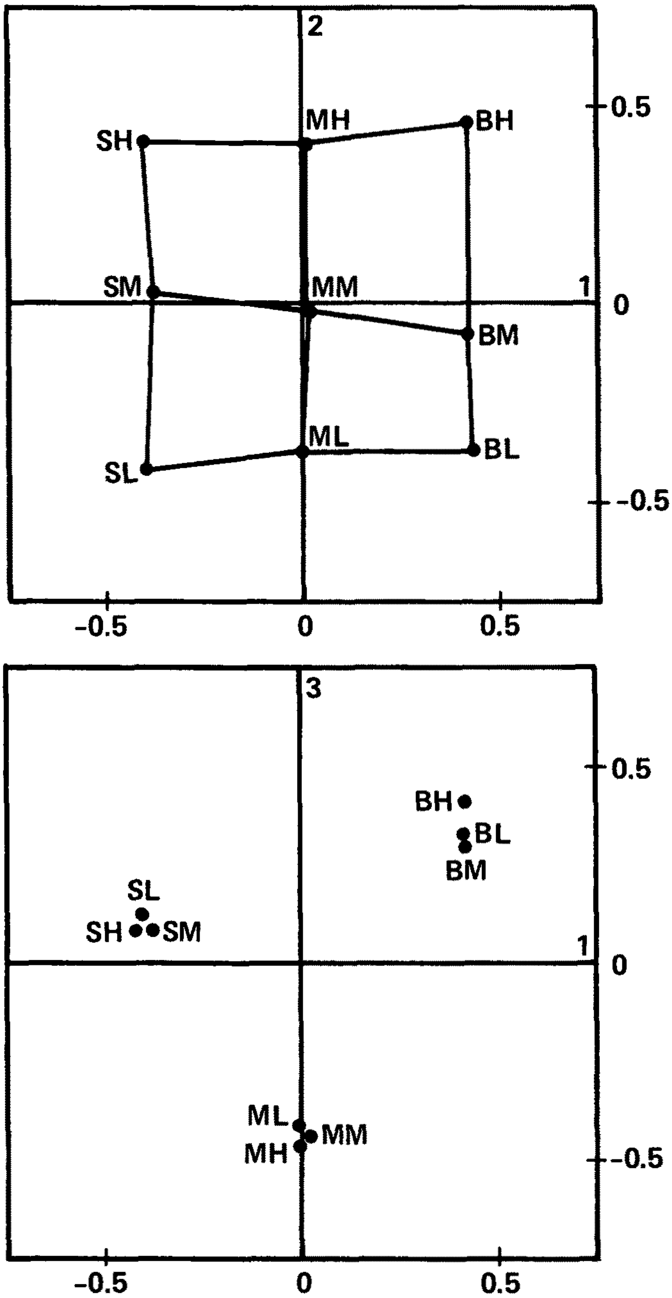

FIGURE 3
Final Transformed Stimulus Space SWI Analysis—Experiments 1 & 2: Dimension 1—Size (1); Dimension 2—Weight (without illusion); Dimension 3—Size (2)

had the desired effect. It is important to note also that for idealized subject 2 the off-diagonal entry is smaller than that for idealized person 1, so that the ratio of the off-diagonal entry to the diagonal entry for the heaviness dimension is almost the same for the two idealized persons. This implies that the strength of the illusion was not modified by the experimental manipulation, (as will be discussed in more detail below). Finally, an interesting point that comes out of the similarity of separate and combined TMMDS analyses is

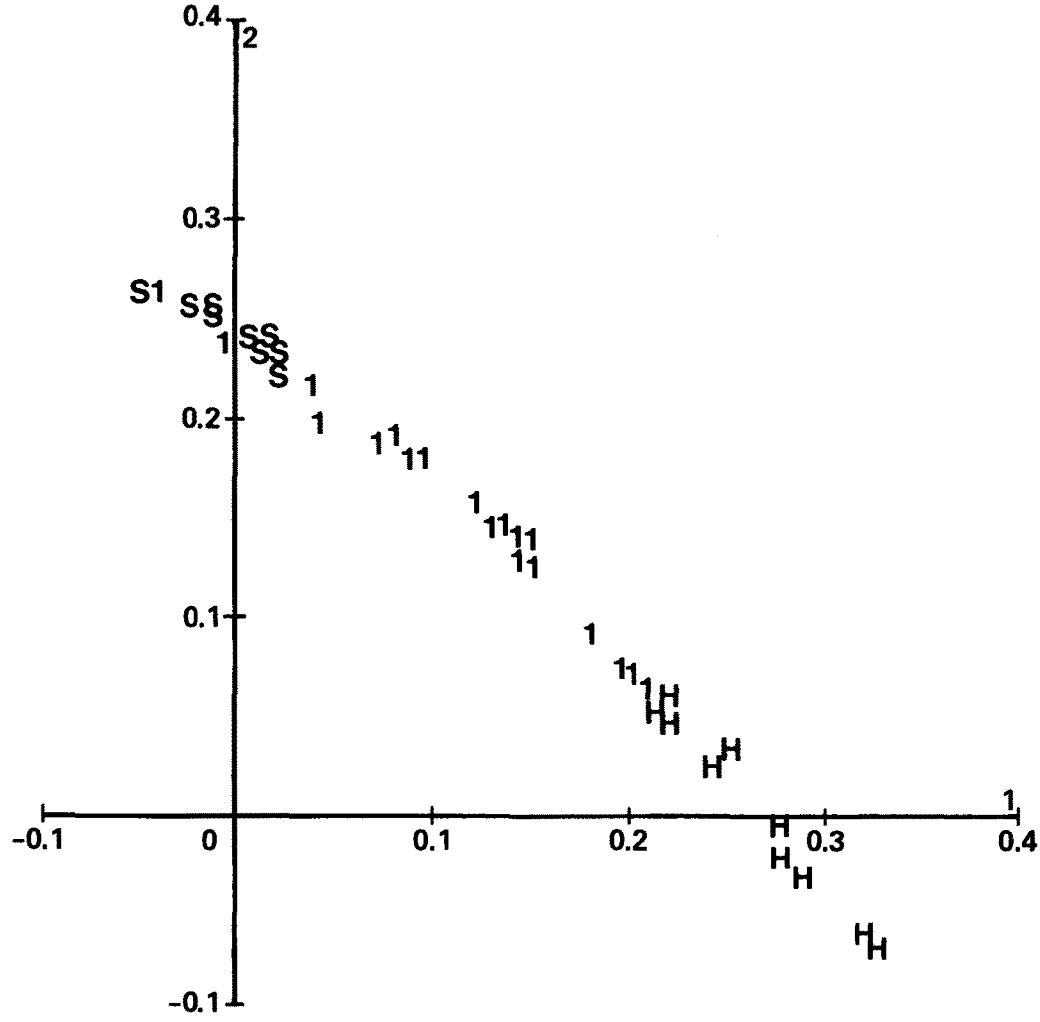

FIGURE 4
Transformed TMMDS Person Space Experiments 1 & 2.

that the behavior of the subjects in Experiment 1 can be represented as a linear combination of the behaviors in Experiment 2.

The person characteristic matrices were then factored according to the SWI model given in (12). The parameters of the model were obtained by the procedure set out in (14) through (17). Table 3 gives the parameters. The superscripts represent the experiment numbers. Thus $SW_{i2}^{(1)}$ stands for the SWI-model weight for person i on dimension 2 in Experiment 1. Subjects 1 through 10 were in Group H and subjects 11 through 19 were in Group S in Experiment 2. The weights and the θ_i varied with the experimental conditions. For Group H subjects, θ_i increases going from Experiment 1 to Experiment 2 as does SW_{i2}, while SW_{i1} and SW_{i3} all decrease. The opposite is true for the Group S subjects with the exception that SW_{i1} and SW_{i3} decrease for subject 16. Thus in terms of the model of the size-weight illusion presented above, the parameters seem to suggest that the instructional conditions had the effect of increasing the influence of the illusion on the subject's ratings when they were told to pay more attention to heaviness, and decreasing the influence of the

TABLE 2

Transformed Core Matrix
Experiments 1 and 2

Idealized Person 1

	1	2	3
1	18.492	-6.479	-0.138
2	-6.479	21.096	0.116
3	-0.138	0.116	5.464

Idealized Person 2

	1	2	3
1	39.700	-2.111	0.102
2	-2.111	7.828	0.007
3	0.102	0.007	12.315

illusion when they were asked to pay more attention to size. This point will be taken up in later discussion.

The perceptual spaces for subjects 1 and 15 in Experiment 2 were computed using (9) and are shown in Figure 5. The projections on the heaviness dimension for stimuli of the same mass decrease with increasing size, reflecting the usual size-weight illusion. The projections on the size axis for stimuli of the same size are approximately equal.

INDSCAL analysis. The data from Experiments 1 and 2 were also analyzed by the INDSCAL procedure. The stimulus space for a three dimensional solution is shown in Figure 6. Stimuli of the same size have approximately equal projections on dimension 1, and the projections on dimension 2 are proportional to "heaviness", i.e., stimuli having the same mass decrease with increasing size. Thus the INDSCAL stimulus space is quite interpretable. In fact, the INDSCAL stimulus space is similar in shape to the SWI perceptual spaces shown in Figure 5. Weighting by the subjects' weights produces INDSCAL perceptual spaces that are quite similar to the SWI weighted perceptual spaces. Table 4 shows the rank order correlations between the person parameters of the SWI and INDSCAL models. These are computed across the 38 observations from the two experiments. The negative correlation between SW_1 and SW_2 resulted from the constraints of the rating scale. If a subject paid more attention to size, it would be at the expense of heaviness. There are other necessary relations among the SWI parameters which arise because the rank of the person space was two.

For dimensions one and two, the relationship between the INDSCAL subject weights and the SWI parameters are as would be expected given the similarity of the stimulus

TABLE 3

Parameters for the Size-Weight Illusion
Experiments 1 & 2

Subject	$\theta_i^{(1)}$	$\theta_i^{(2)}$	$SW_{i1}^{(1)}$	$SW_{i1}^{(2)}$	$SW_{i2}^{(1)}$	$SW_{i2}^{(2)}$	$SW_{i3}^{(1)}$	$SW_{i3}^{(2)}$
1 (20)	16.326	17.212	2.701	1.777	1.825	2.681	1.524	1.041
2 (21)	15.687	16.988	2.702	2.249	1.452	2.396	1.519	1.287
3 (22)	16.658	17.087	2.573	2.077	2.078	2.508	1.458	1.197
4 (23	16.850	16.952	2.424	2.336	2.256	2.395	1.379	1.334
5 (24)	16.648	17.148	2.582	1.992	2.072	2.645	1.463	1.155
6 (25)	16.930	17.259	2.353	1.652	2.358	2.797	1.343	.979
7 (26)	16.690	17.278	2.563	1.576	2.115	2.823	1.453	.940
8 (27)	16.120	17.004	2.715	2.224	1.684	2.413	1.529	1.274
9 (28)	16.575	17.060	2.637	2.201	2.024	2.560	1.492	1.265
10 (29)	16.921	17.191	2.341	1.827	2.325	2.634	1.335	1.066
11 (30)	16.164	14.517	2.718	2.833	1.715	1.176	1.531	1.589
12 (31)	16.945	14.843	2.344	2.828	2.384	1.246	1.338	1.586
13 (32)	16.682	15.119	2.552	2.811	2.094	1.311	1.446	1.578
14 (33)	15.490	12.708	2.785	2.844	1.418	.924	1.564	1.593
15 (34)	14.000	11.053	2.798	2.855	1.069	.797	1.568	1.598
16 (35)	1.661	-14.729	2.858	2.827	.506	.364	1.599	1.583
17 (36)	16.603	14.991	2.618	2.821	2.043	1.280	1.482	1.583
18 (37)	16.542	13.271	2.629	2.844	1.981	.986	1.487	1.593
19 (38)	16.286	15.218	2.675	2.799	1.776	1.335	1.508	1.571

spaces obtained by the two procedures. The subject weights for the third INDSCAL dimension show only small correlations with the other subject parameters. This result is explained by the fact that the rank of the INDSCAL matrix of subject weights is three. Thus, in effect, there are more parameters in the INDSCAL solution than in the SWI solution. The inequality of numbers of parameters makes it a little difficult to compare the two models in terms of fit to the data. The correlations between the data and the predictions of the models were .956 for the SWI solution and .965 for the INDSCAL solution; but the slightly better INDSCAL fit may simply reflect the fact that INDSCAL used more parameters. Alternatively, the difference might be due to the fact that the INDSCAL algorithm provides a least squares fit while the TMMDS algorithm used to fit the SWI model provides a fit which is only approximately least squares. In any case the differences are quite small, and in general we can conclude that both models provide a good fit to the data.

Discussion

Analysis Using the SWI Model

This example has demonstrated how a special purpose model, such as the SWI model, can provide a means of dealing effectively with the indeterminacies of the TMMDS or

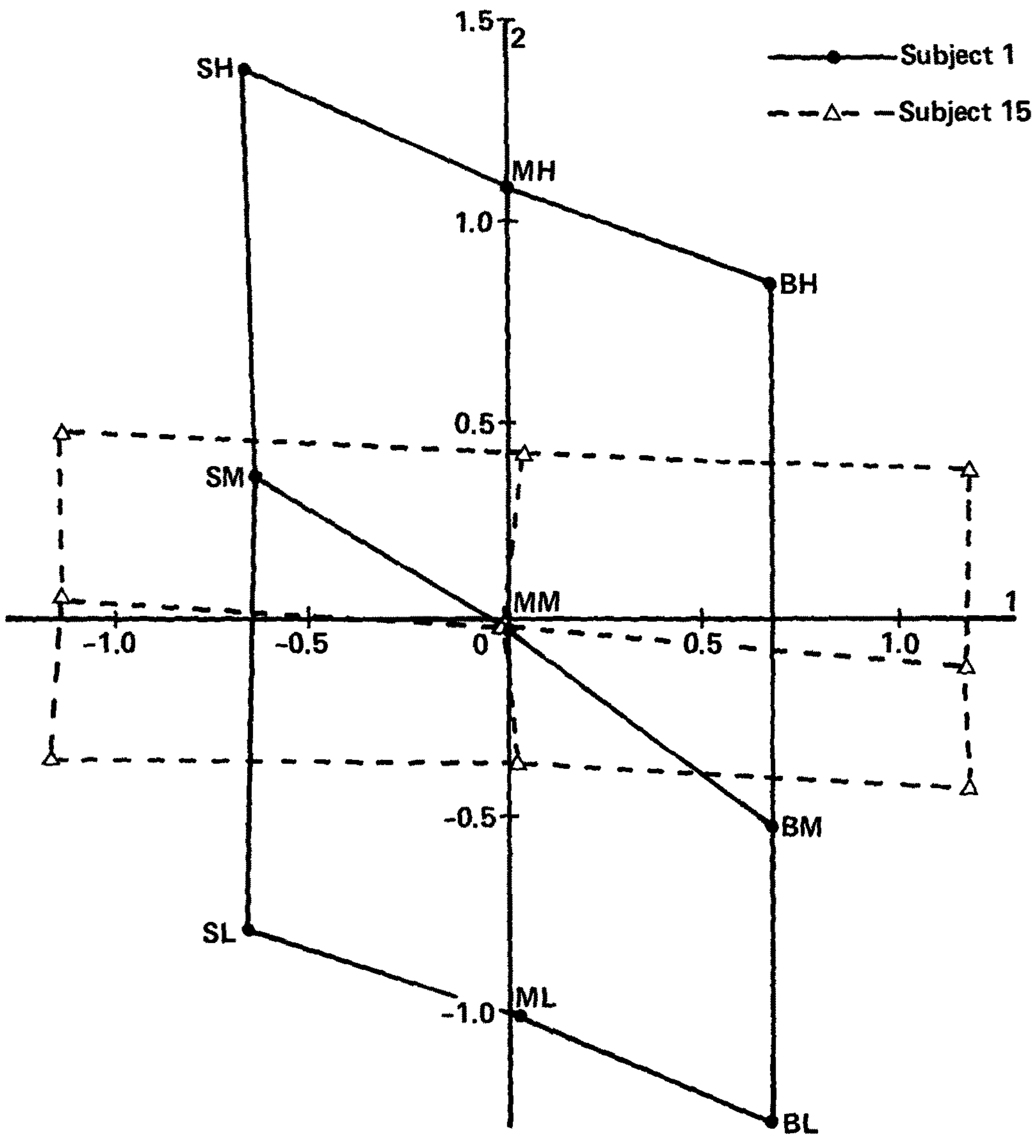

FIGURE 5
Perceptual Spaces for Selected Subjects Experiments 1 & 2: Dimension 1—Size; Dimension 2 Heaviness with Illusion

IDIOSCAL solutions. We began with a TMMDS analysis; from the initial, arbitrary, TMMDS solution, the SWI model led us directly to a meaningful stimulus space and person space, as well as a clearly interpretable core matrix. By computing person characteristic matrices $\mathbf{H}_i$ and transforming them to the special form of (12), we were able to obtain parameters that expressed the strength of the illusion for each subject. These parameters are harder to obtain in the weighted euclidean representation. Although there is a way of representing individual differences in illusion strength in terms of the weighted euclidean model, this would require an additional dimension which is linearly dependent on the first two in the stimulus space [Harshman, Note 2]; this approach does not seem advantageous, since it would reintroduce indeterminacies into the solution. The SWI model provides a

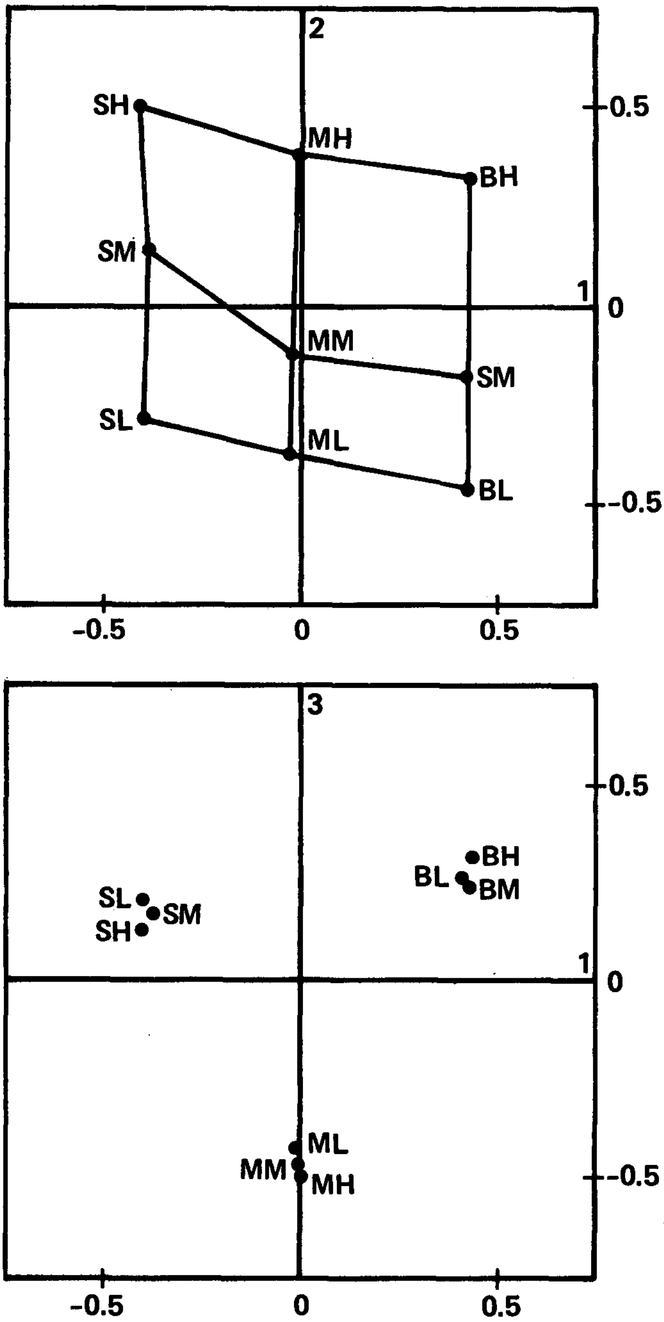

FIGURE 6
INDSCAL Stimulus Space Experiments 1 & 2: Dimension 1—Size (1); Dimension 2—Heaviness (with illusion); Dimension 3—Size (2)

viable method of maintaining the extra generality desired while overcoming the problem of indeterminacy.

Although the SWI model allows for individual differences in the strength of the illusion, it turned out that our analysis indicated only minor variations in the magnitude of the illusion across subjects (with one exception, the outlier subject 35; see line 16 of Table 3).

TABLE 4

Rank Order Correlations Among SWI and INDSCAL Person Parameters - Experiments 1 & 2

	θ	SW_1	SW_2	SW_3	IW_1	IW_2	IW_3
θ	1.000						
SW_1	-.989	1.000					
SW_2	1.000	-.989	1.000				
SW_3	-.988	1.000	-.987	1.000			
IW_1	-.962	.969	-.961	.971	1.000		
IW_2	.998	-.988	.999	-.986	-.958	1.000	
IW_3	-.278	.274	-.278	.273	.128	-.292	1.000

Although these variations were moderately reliable (the correlation between $\theta_i^{(1)}$ and $\theta_i^{(2)}$ within each experimental group was .44 for Group H and .75 for Group S, even with the outlier omitted), they were nonetheless quite small. Excluding the outlier subject, $\theta_i^{(1)}$ has a range of approximately $\pm$ 1.5 degrees, and a standard deviation of less than 0.7 degrees. Even with the manipulations of Experiment 2, the standard deviation of $\theta_i^{(2)}$ is only 1.8 degrees. Such consistency was not expected prior to performing the analysis.

Analysis Using the Weighted Euclidean Model

Despite its lack of generality, the weighted euclidean model also produced a highly interpretable (although somewhat less informative) solution. In fact, the SWI and INDSCAL solutions were in most respects quite similar; both revealed a size and heaviness dimension, and the subject weights for both solutions were highly correlated. However, the weighted euclidean model has no means (within the three dimensions considered) of representing individual differences in the magnitude of the illusion. Consequently, the INDSCAL stimulus space can be considered a transformation of the SWI theoretical space shown in Figure 1, which uses a single fixed θ to approximate the various θ_i of the SWI model. To make explicit what this single θ would be, we transformed the SWI theoretical space via an oblique procrustean rotation, using the first two dimensions of the INDSCAL stimulus space as a target. This procedure produced a transformation matrix which approximates that shown in (7), and for which the constant angle was 16.2 degrees.

If there had been substantial individual differences in the strength of the illusion, these data might have provided an interesting test of the ability of INDSCAL to deal with violations of the weighted euclidean model. One wonders whether we would have obtained results similar to those of MacCallum [1976a] who found that oblique perceptual dimensions (a different but related form of violation of the weighted euclidean model) could cause the INDSCAL solution to be distorted, and could give rise to negative subject weights. He was only able to demonstrate these effects, however, with synthetic data.

MacCallum introduced into his synthetic data large individual differences in obliqueness among dimensions that were independent of individual differences in weights applied to dimensions. This required that the rank of the person space be three. In our size-weight illusion data, however, the individual differences in the influence of the illusion were rather small and the rank of the person space was two. Furthermore, since the two planes of the core matrix were both positive definite, a unique transformation matrix could be found which would simultaneously diagonalize both planes of the core matrix, and hence an INDSCAL solution existed. (In MacCallum's synthetic data, where the dimensionality of the person space was three, an INDSCAL solution did not exist.) Consequently, the lack of negative subject weights or other anamolies with this data is to be expected, and tells us little about the behavior of INDSCAL with other data which might more seriously violate the assumptions of the weighted euclidean model.

Comparison of the Two Solutions

The SWI and the INDSCAL solutions provide similar descriptions of the perceptual dimensions employed by our subjects. In one sense, then, INDSCAL confirms the psychological theory behind the SWI model: it indicates that the directions in the common perceptual space in which subjects tended to stretch or contract subjective distances correspond to the directions that would be predicted for such stretches or contractions on the basis of our theory of the size-weight illusion. Of course, given the simple psychological situation involved, this is not too surprising. Still, this verifies that size and heaviness (under the influence of the illusion) were indeed the perceptual dimensions used by the subjects in making their dissimilarity judgments.

As we have already noted, there is a sense in which the SWI solution can be said to have been more informative than the INDSCAL solution. Only the SWI model allowed us to examine the individual differences in the strength of the illusion. Moderately reliable individual differences were, in fact, found, and in some psychological contexts these differences might prove to be interesting (e.g., as correlates of other individual differences, such as differences in cognitive style, susceptibility to the Muller-Lyer illusion, etc.). However, it was interesting to discover that in the experimental situation used here, the differences in strength of the illusion were quite small. The INDSCAL solution would not have permitted us to make this observation.

There is a further reason that the SWI model might be preferred. The derivation of the SWI model involves a theory of the perceptual process by which heaviness is derived from size and weight. Thus the SWI analysis provides a test (or at least an application) of this theory. Indeed, the SWI model is simply an extension of the additive contrast model of the size-weight illusion presented by Anderson [1970] and Birnbaum and Viet [1974]. In fact, the Anderson model can be considered as a special case of the SWI model in which all the subject weights for the size dimension are zero. In this way, the SWI model can represent subjects' judgments of dissimilarity even when size is not explicitly used.

Other Issues

Oblique vs. orthogonal axes. The results of this study in part support Harshman's [Note 4] finding that the size-weight illusion induces an oblique relationship between size and weight in the subject's perceptual space. However, the results reported here differ in two respects from Harshman's results. First, the SWI analysis indicated only small variations in the angle between size and weight across subjects (with the exception of one outlier subject). Harshman reported "a wide range of individual variations in the angle between perceptual dimensions" (p. 51). Second, the SWI model proposed here assumes an orthogonal perception of size and heaviness dimensions, as implied by (10). Harshman found evidence of oblique perceptions of size and heaviness, as well as size and weight, in many of his subjects. Specifically, when he used regression to place unidimensional scales of size and heaviness into each subject's perceptual space, the angle between the size and heaviness axes differed reliably from zero (for many subjects).

There are, however, important methodological differences between the present study and the earlier study by Harshman. In Harshman's study, the weights being compared were lifted sequentially using only one hand. And although the subject was free to go back and forth between the two weights, (repeatedly lifting until he was confident of his dissimilarity judgment), the actual comparison was made in memory. This might have allowed a greater scope for subjective individual differences. Also, Harshman used a broader range of sizes and weights, so that subjects would be less likely to recognize a standard size and try somehow to standardize the response. Finally, Harshman directly measured unidimensional scales of size, heaviness, and weight, for each subject; he also performed a separate multidimensional scaling of each subject's dissimilarity judgments, to obtain for each subject an independent perceptual space. He then used regression to fit a given subject's unidimensional scales into that subject's multidimensional space. In the present study, no attempt was made to directly measure the unidimensional scales for size, weight, and heaviness. These were inferred from the multidimensional scaling analysis. Also, a common perceptual space was estimated for all subjects, allowing only those patterns of variation which were consistent with the three-way SWI model (e.g., the perceptual spaces of all subjects were constrained so that size and heaviness would be orthogonal; this restriction might have considerably reduced the individual variations in angles between size and weight). It is possible that one or more of these methodological differences may have caused the different results in the two studies.

Other aspects of the SWI solution. It appears from Table 3 that the manipulation of subjects' attention, (by directing group H to pay more attention to heaviness, and group S to pay more attention to size), has an effect on the strength of the size-weight illusion. Subjects asked to pay more attention to heaviness showed a greater illusion than those asked to pay more attention to size, since the $\theta_i^{(2)}$ values in group H tend to be larger than those in group S. However, the effect is modest in size, and might be due to characteristics of the estimation procedures used in fitting the SWI model. If, for example, the effects of individual differences in dimension weights were not perfectly estimated and completely removed from each individual's space, then a relationship such as the one observed would be likely to occur as an artifact. Subtle aspects of the SWI solution, such as this apparent effect of experimental instructions on θ_i, should be checked in some independent fashion before they are taken as firm evidence of a psychological effect. Similarly, the perfect rank order correlation between θ_i and SW_2 (Table 4) might at first seem to indicate a striking psychological relationship. However, there is an alternative explanation.

If the θ_i in the SWI analysis were to be exactly constant, the condition that would need to be met is that the ratios of the off diagonal to the diagonal elements for weight in the two planes of the core matrix would need to be exactly equal. These two ratios were $-.31$ and $-.27$ for idealized persons 1 and 2 respectively. Thus, the perfect rank order correlation in Table 4 between θ_i and SW_2 arises because the larger ratio occurs for the idealized person having the larger diagonal element for weight, and hence is probably artifactual.

Constrained analyses. Although we have conceptualized our approach as one of fitting a model of intermediate generality between the weighted euclidean and the TMMDS or IDIOSCAL, in fact we have fit the TMMDS model and then used procrustes rotation based on the SWI model to transform the solution into maximum conformance with the SWI model. This shows how an *a priori* model can be used to deal with the indeterminacies of the most general three-way MDS procedures. If we wished to fit a more constrained model to the data directly, we might consider modified estimation procedures similar to those developed by Bloxom [1978], and Carroll, Pruzansky and Kruskal [1980] which show how linear constraints can be placed on the projections in the object space. For example, constraints could presumably be incorporated into TMMDS procedures by methods similar to those used by Carroll, et al. for the weighted euclidean model. Such methods might then be employed, for example, to insure that all stimuli of the same size have identical projections on the size dimension. Comparison of fit values for the constrained and unconstrained solutions should provide further information on the appropriateness of the SWI model (or whatever *a priori* model one is fitting) for a given set of data. In any case, the use of constrained as opposed to unconstrained solutions would not change the basic points made in this article.

Conclusion

This study proposed a model of the size-weight illusion approached from the viewpoint of individual differences models in multidimensional scaling. By specifying in advance the stimulus space and the decomposition of the individual characteristic matrices, it was possible to eliminate the indeterminacies and problems of interpretation associated with models such as IDIOSCAL and TMMDS. This procedure was applied to data from two experiments and compared to an INDSCAL analysis of the same data.

The size-weight illusion model provided a more informative solution, because it allowed an evaluation of the individual differences in the strength of the illusion. However, because such individual differences turned out to be minor in these experiments, the INDSCAL analysis was able to produce an interpretable solution which ignored inter-subject differences in magnitude of the illusion.

The size-weight illusion study provides an example of how "special purpose" theoretical models tailored to represent the psychological processes under investigation can provide an effective way of examining individual differences more general than those allowed by the weighted euclidean model, while avoiding the difficulties sometimes associated with the indeterminacies of the very general IDIOSCAL and TMMDS models.

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Manuscript received 10/18/77
First revision received 5/11/79
Second revision received 9/12/80
Final version received 9/25/81