Multivariate Behavioral Research, 29 (1), 63-113

Structural Equation Models of Factorial Invariance in Parallel Proportional Profiles and Oblique Confactor Problems

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Some problems of multiple group factor rotation based on Cattell's "parallel proportional profiles" and "confactor rotation" are described (see Cattell, 1944, 1966, 1972). Some relations between these classic ideas and contemporary practices in structural equation modeling (e.g., LISREL) are explored. We show how the Confactor approach: (a) is related to Meredith's (1964a) selection model, (b) can be a parsimonious model for multiple group factor analyses, and (c) how this model can be fitted using standard structural equation modeling techniques. We discuss several alternative structural modeling solutions, including (d) selection of a good reference variable solution, (e) rotation of the invariant orthogonal structure by standard rotation routines, and (f) higher-order, latent paths, and latent means structural model restrictions. Mathematical and statistical properties of these models are examined using Meredith's (1964b) four group problem fitted by Jöreskog and Sörbom's (1979, 1985) LISREL algorithm. The benefits and limitations of this structural modeling approach to oblique Confactor resolution are examined and opportunities for future research are discussed.

Introduction

Structural equation models may be evaluated in terms of both parsimony and accuracy. We often index the complexity or simplicity of a set of

Prepared for publication in *Multivariate Behavioral Research*, September 1987; Accepted November 1989; Final revisions March, 1993. The ideas of this article were originally developed and discussed with several other researchers. Especially we thank John Horn, Bill Meredith, John Nesselroade, Jerry Brennan, and Michael Browne for the free exchange and use of their ideas. We are indebted to Robert Cudeck for his editorial comments, and to three unnamed reviewers for helpful suggestions. This work was reported at the Annual Meeting of the *Society of Multivariate Experimental Psychology*, Charlottesville, Virginia, November 1988. This research has been supported by grants from the National Institute on Aging (AG07137). Requests for reprints should be sent to the first author at the Department of Psychology, University of Virginia, Charlottesville, VA 22903.

structural equation models by counting the number of parameters (or degrees-of-freedom) required by different models for the same data. We often index the accuracy of the model predictions by using the likelihood ratio test statistics (e.g., χ^2) for models fit to covariance, cross-products, or to raw data. These issues naturally arise in fitting factor analysis models. In a given set of data, a one-factor model is often a simpler and more parsimonious representation than a two-factor model, but the two-factor model may fit the observed data better. Relationships between parsimony and accuracy are fundamental aspects in the choice between different models.

In structural equation models with multiple groups we often require invariance of some model parameters, and this adds another level of complexity to our decisions. In the case of common factor models, for example, it is not clear whether the requirements of a *simple structure* in a factor pattern should be more or less important than the requirements of *factorial invariance* over groups. Parsimony in multiple groups models may be based on comparisons of the benefits of invariance over groups against the costs of complexity of a factor pattern. Accuracy of model fit is now based on how well we account for the data of several independent groups. These parsimony versus accuracy tradeoffs are crucial in multiple group models.

In this research we revive the seminal ideas of the Parallel Poportional Profiles or Confactor model originally stated by Cattell (1944, 1966, 1972; Cattell & Cattell, 1955). We also highlight issues of factorial invariance under selection detailed by Meredith (1964a, 1964b, 1965, 1990). We then restate these model principles using the standard techniques of linear structural equation modeling (e.g., LISREL by Jöreskog & Sörbom, 1979, 1985; COSAN by McDonald, 1980, Fraser, 1979). This merger leads us to examine multiple groups factor models with *factorial invariance as the primary consideration and simple structure as a secondary consideration.* We point out mathematical conditions needed for unique identification of multiple group model parameters, suggest a few practical solutions for the unique rotation of a factorial invariant model, and highlight alternative ways to search for a potentially accurate and parsimonous structural models.

Simplicity and Factorial Invariance

The principles of "simple structure rotation," originally defined by Thurstone (1935, 1947) are well known and widely used in all forms of multiple factor analysis. Thurstone based simple structure on a principle of invariance:

> One of the turning-points in the solution of the multiple factor problem is the concept of "simple structure." It will be shown that this concept enables us to obtain an invariance of factorial description that has not, so far, been available by other means. ... When a factor matrix reveals one or more zeros in each row, we can infer that each of the tests does not involve all the common factors that are required to account for the intercorrelations of the battery as a whole. This is the principle characteristic of a simple structure (Thurstone, 1947, p.181). ... *the factorial description of a test must remain invariant when the test is moved from one battery to another which involves the same common factors* (Thurstone, 1935, p.120; 1947, p.361). *The factorial composition of a set of primary factors that have been found in a complete and overdetermined simple structure remains invariant when the test is moved to another battery involving the same common factors and in which there are enough tests to make the simple structure complete and overdetermined* (Thurstone, 1947, p.363).

Thurstone (1947, p.365) also distinguished "configurational invariance", or the invariance of the zero loadings, from "metric invariance", or the invariance of the numerical values of all loadings (see Horn, McArdle & Mason, 1983; Horn & McArdle, 1992). A review of the recent literature on confirmatory factor analysis shows Thurstone's seminal principles of simple structure remain important today. Indeed, it seems fair to say that this simple structure principle has been the main theme of confirmatory factor model analyses. Unfortunately, the basic requirement of a "complete and overdetermined" sampling of variables is not often available in these structural modeling experiments (see McDonald, 1985; Horn, 1988).

There have been relatively few alternatives to the simple structure principles. However, under the heading of "The Most Fundamental Principle," Cattell (1944) offered such an alternative:

> ... The principle of parsimony, it seems, should not demand "Which is the simplest set of factors for reproducing this particular correlation matrix?" but rather "Which set of factors will be most parsimonious at once with respect to this and other matrices considered together?" ... The criterion is then no longer that the rotation shall offer the fewest factor loadings for any one matrix; But that it shall offer fewest dissimilar (and therefore fewest total) loadings in all of the matrices together. ... To indicate the historical foundations from which it builds, however, and the fact that it extends to several matrices simultaneously the principle of parsimony involved in simple structure, it might equally well be called 'simultaneous simple structure (pp.273-274).

Cattell's theoretical model was initially designed to reflect substantive concerns:

> The basic assumption is that, *if a factor corresponds to some real organic unity, then from one study to another it will retain its pattern, simultaneously raising or lowering all of its loadings according to the magnitude of the role of that factor under different experimental conditions of the second study.* No inorganic factor, a mere mathematical abstraction, would behave this way, within the realm of orthogonal factors. The principle suggests that every factor analytic investigation should be carried out on at least two samples, under conditions differing in the *extent* to which the same psychological factors (working as independent, orthogonal influences) might be expected to be involved. We could then anticipate finding the 'true' factors by locating the unique rotational position (simultaneously in both studies) in which the factor of the first study is found to have loadings which are proportional to (or some simple function of) those in the second: that is to say, a position should be discoverable in which the factor in the second study will have a pattern which is the same as the first, but it is stepped up or down (Cattell & Cattell, 1955, p.84).

Cattell initially believed that invariance of the factor loadings was by itself a criterion that led to a unique rotation of factors. Unfortunately, these ideas proved difficult to demonstrate and prove. Cattell (1944) demonstrated an orthogonal solution to a proportional profiles between two factor positions that was unique, and he found this position by trial and error. Cattell and Cattell (1955) showed that an analytic solution was possible for the two-group orthogonal model and used the term "Confactor" rotation for this solution. Cattell (1966) summarized the gains to date, worked with covariances, and explored avenues towards the oblique solutions. One of the latter was by the Schmid-Leiman transformation and involved the idea that the condition of proportionality (parallel proportional profiles) must be extended to higher order factors. Cattell and Brennan (1977) showed how the orthogonal solution could be used as a first approximation to the oblique solution, but also how this failed to obtain a unique oblique position. More recently, McArdle (1984a) and McArdle and Cattell (1988) showed how a structural modeling approach demonstrated both limitations and benefits of the oblique Confactor model. Details of these structural Confactor solutions will be presented here.

Selection and Factorial Invariance

Some important mathematical work has been directed at the Confactor problem. Prominent among these is the research of Meredith (1964a, 1964b, 1965). Meredith followed theorems by Pearson (1903), Aitken (1934), and Lawley (1943), and applied basic mathematical principles of multivariate selection to the multiple factor invariance problem:

> Cattell [1944] has enunciated a principle of rotation called 'parallel proportional profiles' that bears an interesting relationship to the present results. One of the situations proposed by Cattell as a case in which the principle of parallel proportional profiles could be applied is the alteration of the population, i.e., selection of subjects or, in our terminology, examination of different subpopulations. The principle of parallel proportional profiles, applied in this case, states that, given two factor pattern matrices obtained from two populations derived from a common parent by selection, there ought to exist a rotation of the two matrices so as to make corresponding columns of the rotated factor matrices proportional (Meredith, 1964a, p.182).

Using this approach, Meredith made several practical proposals for the unique resolution of the Confactor problem, clarified several of Cattell's earlier statements, but also raised doubts about the possibility of any unique Confactor solution for the oblique case:

> Our results indicate that 'parallel proportional profiles' cannot be achieved unless each manifest variable is expressed in the same unit of measurement across subpopulations. Furthermore, a stronger result, namely invariance of the factor pattern matrix, can be obtained if we are not required to express the factor variables with unit variance in each subpopulation. However, such a result is not unique; hence in general there will exist an infinite number of matrices satisfying the principle of 'parallel proportional profiles' for any pair of populations derived by selection satisfying the requirements of Lawley's Theorem. If, however, we require that for a given pair of subpopulations the factor variables be uncorrelated in each, a unique result is obtained. ... given any pair of populations derived by selection on y, there exists an orthogonal factor pattern matrix such that the corresponding orthogonal factor structure matrices satisfy the requirement of 'parallel proportional profiles' ... This factor pattern is unique to a given pair of populations since the eigenvalues and vectors of $T^{-1}C_2(T')^{-1}$ are unique. The development (27) through (32) is essentially that given by Cattell and Cattell [1955] ... (Meredith, 1964a, p.182-183).

Meredith emphasized the benefits of a selection approach to factorial invariance:

> Lawley's Selection Theorem is really an extraordinarily powerful tool ... This means that the conditions of this theorem hold regardless of the type of selection (i.e., truncated, probabilistic, etc.) and hence that out conclusions about factorial invariance hold regardless of the type of

> selection. Since the only restriction on the distribution of selection variates is the nonsingularity of the variance-covariance matrix in the parent population, the condition of linearity of regression of nonselection on selection variates is almost trivial. The major requirement then becomes the requirement of homscedasticity. Another remarkable result of these findings is that we do not even need to know what the selection variables are, much less be able to observe them, for factorial invariance to hold... (Meredith, 1964a, 184-185).

Meredith (1964b, 1965) explored several formal solutions to these problems, including rotation of several loading matrices to the same target, the factoring of a pooled covariance matrix, and he also provided goodness-of-fit indices. Bloxom (1968a, 1968b, 1972) extended this work and proposed several interesting alternative models. Bloxom (1968a) showed how invariance of the factor score distribution allowed unique (but not invariant) factor loadings together with simple structure. These important demonstrations show conditions under which factorial invariance will not obtain.

Bloxom (1968b) further examined invariance in the context of Tucker's three-mode factor analysis model. Similar ideas were used in the more recent model termed PARAFAC by Harshman (1970, as reported by Harshman & Lundy, 1984). The term PARAFAC was used to designate the "Parallel Proportional Profile" basis of a Multidimensional Scaling Solution. The oblique version of PARAFAC uses the additional constraint of equal factor correlation matrices to obtain unique parameter estimates (see McDonald, 1984, 1985). This equal correlation restriction departs from Cattell's (1944; Cattell & Cattell, 1955) original oblique model, and it is not consistent with Meredith's (1964a) model sampling approach.

Other mathematical approaches to this rotation problem are based on general classes of orthogonal transformations designed to rotate different group factor patterns towards some consistent pattern (e.g., Cliff, 1966; Green, 1952; Schönemann, 1966). In less formal approaches the invariance of the rotated factor pattern is judged by some form of the congruence coefficient (e.g., Brokken, 1983; Cattell, 1978; Korth, 1978; Nesselroade & Baltes, 1970; Ten Berge, 1977; Walkey & McCormick, 1985). Further theoretical work on the topic is also presented by many others (e.g., Ahmavarra, 1954; Bechtoldt, 1974; Butler, 1969; Overall, 1964).

Structural Modeling and Factorial Invariance

Mutliple group factor analysis are now often fitted using a structural equation approach first detailed by Jöreskog (1971). In this work Jöreskog extended his work on confirmatory factor analysis by developing and demonstrating a computer program for the simultaneous factor analysis from multiple groups. The key feature of this approach was that parameters of the matrices of any group could be (a) *free* to be estimated, (b) *fixed* at prescribed values, or (c) forced to be *equal* to other parameters in any group. Jöreskog (1971) also used key aspects of Meredith's (1964a) theorems and data. In these first structural equation models an oblique factor model was used as a starting point and many additional factor loadings were "fixed at zero by hypothesis." This confirmatory approach is consistent with the simple structure ideas and leads to "over-identified" parameter estimates, less rotational indeterminacy, and numerous degrees-of-freedom to test the basic model hypothesis. This general approach remains important because it allows factorial invariance over groups to be both estimated and tested as a statistical hypothesis.

More complex structural equation models have been dealt with in the recent literature. Bloxom (1972) showed how Jöreskog's (1971) approach could be used to extend Meredith's selection ideas in a variety of different ways. Sörbom (1974, 1978) expanded these models to include latent variable means and path models. These basic ideas have also been discussed by, Alwin and Jackson (1980), Horn, et al. (1983), Nesselroade (1983), McDonald (1984, 1985), and Horn and McArdle (1992), among many others. Some of these issues can be summarized with the aid of the path diagrams of Figures 1 and 2.

Figure 1 illustrates a simple two common factor model for six variables and two groups. In Group 1 the factors have a simple pattern of factor loadings (labelled $\boldsymbol{\lambda}_{m,k}{}^{(1)}$), where each variable loads on one and only one factor. This group also has a correlation between the two factors (labelled $\boldsymbol{\rho}_{1,2}{}^{(1)}$), and a vector of uniqueness (labelled $\boldsymbol{\psi}_{m,m}{}^{(1)}$). The variances on both factors have been set equal to one for the purposes of model identification. In Group G we find the same two factors, the same configuration of factor loadings (labelled $\boldsymbol{\lambda}_{m,k}{}^{(G)}$), a correlation between the two factors (labelled $\boldsymbol{\rho}_{1,2}{}^{(G)}$), and a vector of uniqueness (labelled $\boldsymbol{\psi}_{m,m}{}^{(G)}$). The parameters in Group 1 are labelled with a superscript "(1)" indicating that these parameters are not equal to the parameters with a superscript "(G)". This model is labelled here as having *configural invariance* because the pattern of the loadings of the two factors are the same, but the numerical values of these loadings are not necessarily identical.

Figure 2 illustrates a slightly different two common factor model for the same basic problem. In Group 1 the factors have what appears to be a more complex pattern of factor loadings (labelled $\boldsymbol{\lambda}_{m,k}$); here most variables have loadings on both factors. This group also has a correlation between the two

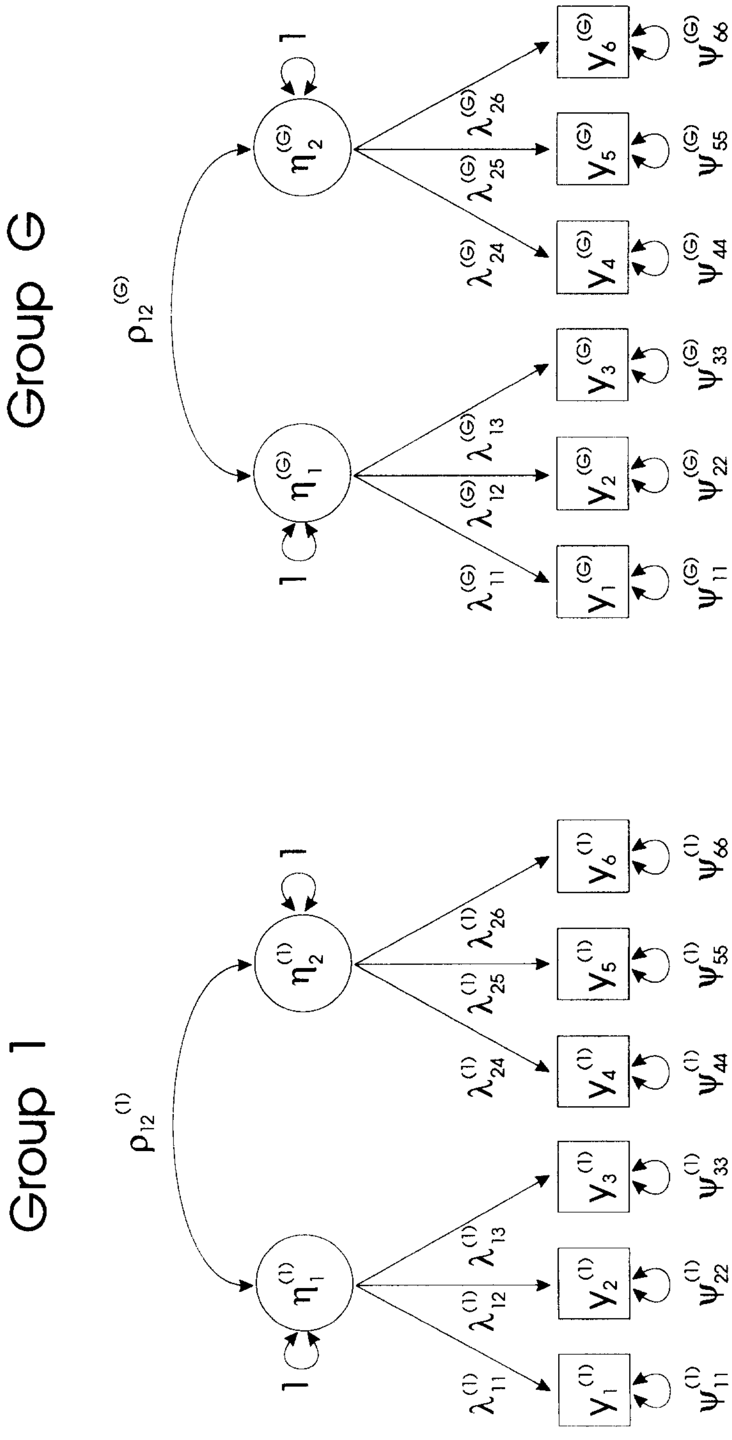

Figure 1
A Two Common Factor Model with CONFIGURAL INVARIANCE over Multiple Groups

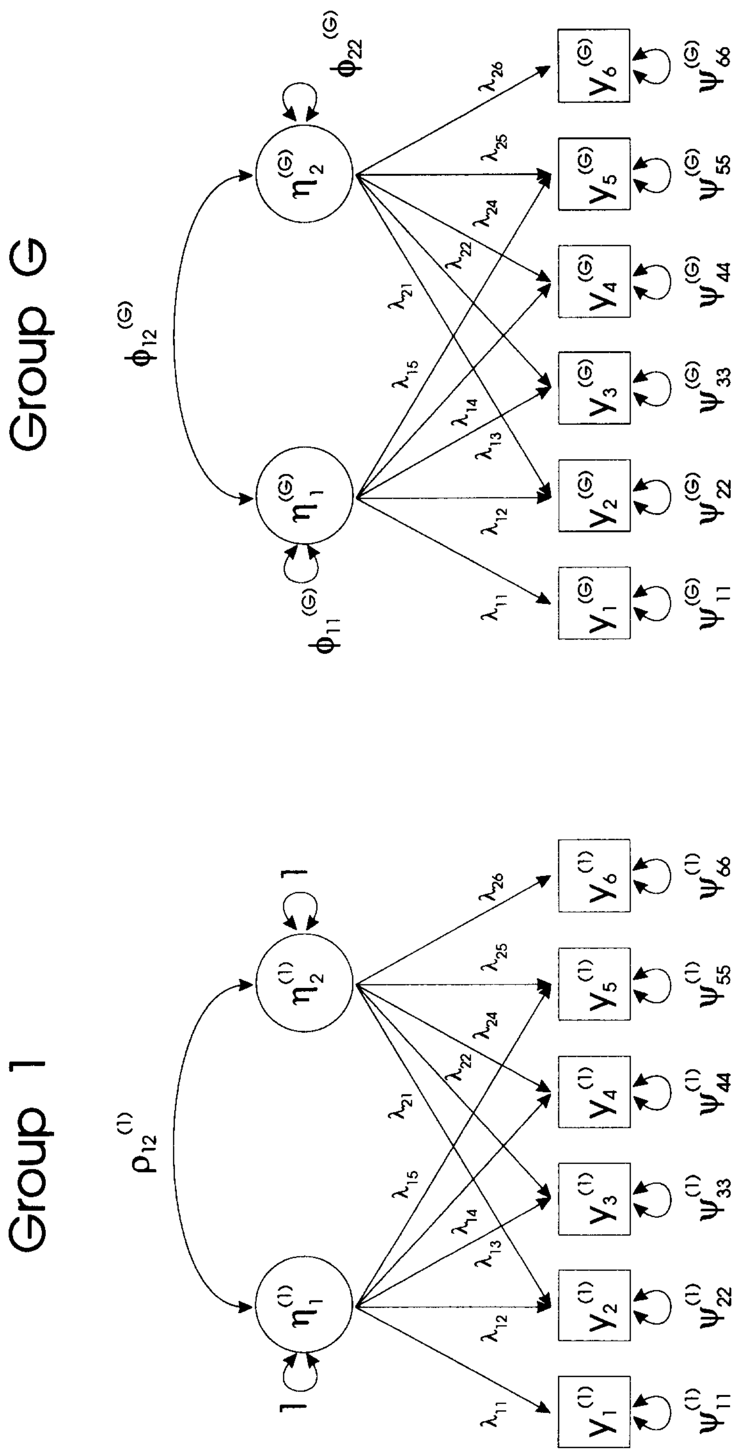

Figure 2
A Two Common Factor Model with CONFACTOR INVARIANCE over Multiple Groups

factors (labelled $\rho_{1,2}^{(1)}$), and a vector of uniqueness (labelled $\psi_{m,m}^{(1)}$). In Group 2 we draw the same two factors with exactly the same complex configuration of factor loadings and with exactly the same values of these loadings (labelled $\lambda_{m,k}$ with no superscript). This model also includes parameters which do vary over groups, including a variance on each factor (labelled $\phi_{k,k}^{(G)}$), a covariance between the two factors (labelled $\phi_{1,2}^{(G)}$), and a vector of uniqueness (labelled $\psi_{m,m}^{(G)}$). This model will be referred to here as having *metric invariance* because the pattern of the loadings of the two factors are the same, and the exact values of these loadings are identical. But Figure 2 has both (a) metric invariance and (b) an exactly-identified factor pattern: We refer to this kind of a model as having *confactor invariance.*

These diagrams demonstrate how apparently similar structural models can be quite different from one another. In the Configural model 1 we have a simple interpretation of the variables in terms of the factors, but the factors are not exactly identical across groups. In the Confactor model 2 we have a complex interpretation of the variables in terms of the factors, but the factors are exactly identical across groups. Given only the two factors and two groups drawn here, models 1 and 2 are both identifiable, and both include the same number of parameters to fit (i.e., 26). However, as we will show later, as the number of groups (G) and number of factors (K) change, the number of parameters needed to define the factors in each model can become substantially different from one another (by G - K). In cases where the number of groups is larger than the number of factors, the Confactor model of Figure 2 can be simpler and more parsimonious than the Configural model of Figure 1. These kinds of model comparisons will be defined, illustrated, and discussed in the rest of this article.

Methods of Structural Confactor Analysis

In this section we detail some basic mathematical requirements of model identification with multiple groups. We assume some algorithm (e.g., LISREL) can be used to estimate the parameters of a common factor model for one or more groups. We define the minimum constraints needed to identify these parameters, and we interpret these constraints as restrictions on transformation matrices required to produce unique factor rotations. We present the original two group orthogonal Confactor solution of Cattell and Cattell (1955) and Meredith (1964a), and we show how the can be expanded for use with oblique rotations. These ideas lead to some novel possibilities for the unique estimation of a partially oblique Confactor model using multiple groups and different rotation criteria. Some practical guidelines for a full structural Confactor analysis are presented.

Single Group Common Factor Models

Let us say we have measured a set of M variables on a group of N individuals. We can now write a structural equation model with K common factors as

$$(1)\qquad \begin{aligned} \mathbf{Y}_n &= \mathbf{\Lambda}\,\boldsymbol{\eta}_n + \boldsymbol{\varepsilon}_n\,, \\ \mathbf{\Sigma} &= \mathbf{\Lambda}\,\mathbf{\Phi}\,\mathbf{\Lambda}' + \mathbf{\Psi}, \text{ and} \\ \mathbf{\Phi} &= \mathbf{\Delta}\,\mathbf{R}\,\mathbf{\Delta}, \end{aligned}$$

where, for individual n, $\mathbf{Y}$ is the M-dimensional vector of observed scores, $\boldsymbol{\eta}$ is the K-dimensional vector of common factor scores, and $\boldsymbol{\varepsilon}$ is the M-dimensional vector of unique factor scores. We further assume $\mathbf{\Lambda}$ is the $(M \times K)$ matrix of common factor loadings, $\mathbf{\Phi}$ is the $(K \times K)$ matrix of common factor covariances, and $\mathbf{\Psi}$ is the $(M \times M)$ diagonal matrix of unique variances. To simplify later notation we also define $\mathbf{\Delta}$ as the $(K \times K)$ diagonal matrix of standard deviations of the common factor scores, and $\mathbf{R}$ as the $(K \times K)$ matrix of correlations among factor scores. Following current structural modeling terminology (e.g., Jöreskog & Sörbom, 1985; McDonald, 1985) we will pattern the elements of matrices $\mathbf{\Lambda}$, $\mathbf{\Phi}$, and $\mathbf{\Psi}$ to be *fixed, free, or equal* as specified by hypothesis. (We will deal with patterning of the mean vector in a later section of this article.)

The estimation of K common factors from a set of M variables requires specific constraints in the factor loadings $\mathbf{\Lambda}$. In the typical unrestricted factor model we initially estimate a K-factor orthogonal solution using some "convenient restrictions" (e.g., such as requiring the initial $\mathbf{\Phi}_i = \mathbf{I}$ and $\mathbf{\Lambda}_i\mathbf{\Psi}^{-1}\mathbf{\Lambda}_i'$ as diagonal; Lawley & Maxwell, 1963; Jöreskog, 1971, p.23). These model parameters are exactly-identified and require further rotation for interpretation. A similar solution can also be obtained from a restricted structural equation model when we place the minimal constraints necessary for a unique solution. These constraints have been described by Jöreskog (1971):

> Two simple sufficient conditions, as given by Howe [1955], are as follows. In the orthogonal case, let $\mathbf{\Phi} = \mathbf{I}$ and let the columns of $\mathbf{\Lambda}$ be arranged so that, for $s = 1, 2, \ldots, k$, column s contains at least $s - 1$ fixed elements. In the oblique case, let diag $\mathbf{\Phi} = \mathbf{I}$ and let each column of $\mathbf{\Lambda}$ have at least $k - 1$ fixed elements. It should be noted that in the orthogonal case there are $\frac{1}{2}k(k + 1)$ conditions on $\mathbf{\Phi}$ and a minimum of $\frac{1}{2}k(k - 1)$ conditions on $\mathbf{\Lambda}$. In the oblique case there are k normalizations in $\mathbf{\Phi}$ and a minimum of $k(k - 1)$ conditions on $\mathbf{\Lambda}$. Thus, in both cases, there is a minimum of k^2 specified elements in $\mathbf{\Lambda}$ and $\mathbf{\Phi}$... (Jöreskog, 1971, p.24).

These constraints can also be interpreted as conditions needed for a unique rotation of the factors. Let us assume we have imposed the required K^2 constraints and fit the K-factor model to M variables. From a set of initial loadings $\mathbf{\Lambda}_i$ and initial covariances $\mathbf{\Phi}_i$ we can write a rotation of the common factors as

$$
\begin{aligned}
(2) \qquad \mathbf{\Sigma} &= \mathbf{\Lambda}_i \, \mathbf{\Phi}_i \, \mathbf{\Lambda}_i' + \mathbf{\Psi}, \\
&= \mathbf{\Lambda}_i \, (\mathbf{T}\,\mathbf{T}^{-1}) \, \mathbf{\Phi}_i \, (\mathbf{T}^{-1'}\,\mathbf{T}') \, \mathbf{\Lambda}_i' + \mathbf{\Psi} \\
&= (\mathbf{\Lambda}_i \, \mathbf{T}) \, (\mathbf{T}^{-1} \, \mathbf{\Phi}_i \, \mathbf{T}^{-1'}) \, (\mathbf{T}' \, \mathbf{\Lambda}_i') + \mathbf{\Psi} \\
&= \mathbf{\Lambda} \, \mathbf{\Phi} \, \mathbf{\Lambda}' + \mathbf{\Psi},
\end{aligned}
$$

where $\mathbf{T}$ is any $(K \times K)$ nonsingular transformation matrix, $\mathbf{\Lambda} = \mathbf{\Lambda}_i\mathbf{T}$ is the rotated factor loading matrix, and $\mathbf{\Phi} = \mathbf{T}^{-1}\mathbf{\Phi}_i\mathbf{T}^{-1'}$ is the rotated factor covariance matrix. If our initial model was orthogonal (i.e., $\mathbf{\Phi}_i = \mathbf{I}$) these same results would obtain. The initial uniquenesses $\mathbf{\Psi}$ are not listed with a subscript because, as shown in Equation 2, these uniquenesses (and communalities) are not changed by the selection of $\mathbf{T}$.

For further clarity, we expand the rotated loading matrix by writing

$$
(3) \qquad \mathbf{\Lambda} = \mathbf{\Lambda}_i\mathbf{T} = \begin{bmatrix} \mathbf{\Lambda}_s \\ \mathbf{\Lambda}_u \end{bmatrix} \mathbf{T} = \begin{bmatrix} \mathbf{\Lambda}_s\mathbf{T} \\ \mathbf{\Lambda}_u\mathbf{T} \end{bmatrix}
$$

where $\mathbf{\Lambda}_s$ is the $(K \times K)$ square submatrix of the initial $\mathbf{\Lambda}_i$ including only the first K rows, and $\mathbf{\Lambda}_u$ is the $[(M - K) \times K]$ non-square submatrix of initial $\mathbf{\Lambda}_i$ including only the remaining $M - K$ rows. There are many choices of the K^2 elements of $\mathbf{T}$, and each of these choices yields a possibly different rotation of the factors defined by loading matrix $\mathbf{\Lambda}$.

To identify this common factor model we initially add K fixed constraints on the scaling of the common factors (i.e., $\mathbf{\Phi}_{k,k} = 1$). The size of this positive value is essentially arbitrary, and these K constraints may be placed elsewhere (e.g., $\mathbf{\Lambda}_s$). An additional $K^2 - K$ restrictions are needed to separate the K factors from one another. In the oblique common factor model this requires all remaining $K^2 - K$ constraints be placed within the factor loading matrix $\mathbf{\Lambda}$. Typically $K - 1$ zero loadings are placed in each of the K columns of $\mathbf{\Lambda}$ by hypothesis. In the orthogonal common factor model we define $K(K - 1)/2$ zero covariances in $\mathbf{\Phi}$ so $K(K - 1)/2$ restrictions are placed within the factor loading elements $\mathbf{\Lambda}$. In either case the location of these constraints is not completely arbitrary: The mathematical side condition required here is the existence of the inverse of a submatrix $\mathbf{\Lambda}_s$ with all fixed values (or the determinant $|\Lambda_s \Lambda_s'| > 0$; Algina, 1980; Shapiro & Browne, 1983; Jöreskog, 1979, p.41, in Jöreskog & Sörbom, 1979). Any set of K^2 *fixed* values of Λ meeting these conditions can be used to identify the K-factor model.

Before we expand this problem to more groups it is useful to highlight one kind of factor rotation. Following the previous equation, we can choose a transformation matrix **T** so

$$
\begin{aligned}
&\mathbf{T} = \Lambda_s^{-1}, \text{ so} \\
(4) \qquad &\Lambda = \Lambda_i \mathbf{T} = \begin{bmatrix} \Lambda_s \\ \Lambda_u \end{bmatrix} \Lambda_s^{-1} = \begin{bmatrix} \Lambda_s \Lambda_s^{-1} \\ \Lambda_u \Lambda_s^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \Lambda_u \Lambda_s^{-1} \end{bmatrix} \text{ and} \\
&\Phi = \Lambda_s \Phi_i \Lambda_s'.
\end{aligned}
$$

In this rotation, the K selected variables have a particularly simple relationship to the common factors: K rows of Λ form an identity (where each variable has unit loadings on one of the first K factors, and zero loadings on each of the other K - 1 factors) and M - K rows of Λ have a possibly complex pattern (defined by $\Lambda_u \Lambda_s^{-1}$). The resulting Φ does not necessarily have unities in the main diagonal, but the necessary scaling of the factors is retained by the unities in the first K rows of Λ. Any subset K of the M observed variables can be selected for inclusion in Λ_s, but each of these selections yields a different ($K \times K$) matrix $\mathbf{T} = \Lambda_s^{-1}$ and corresponding $[(M - K) \times K]$ matrix Λ_u. Following Jöreskog (1971) and Meredith (personal communication, November, 1988) the K observed variables selected for inclusion in the rows of Λ_s will be termed the K reference variables, and we will refer to this procedure as a *reference variable rotation.*

Invariant Factor Loadings in Two Groups

Let us now say we have measured the same M variables on two independent samples of individuals with possibly different sample sizes ($N^{(1)}$ and $N^{(2)}$). We write a model for each group with K common factors and an invariant factor loading matrix Λ as

$$
\begin{aligned}
(5) \qquad &\mathbf{Y}_n^{(1)} = \Lambda\, \eta_n^{(1)} + \varepsilon_n^{(1)}, \quad \Sigma^{(1)} = \Lambda \Phi^{(1)} \Lambda' + \Psi^{(1)} \text{ and} \\
&\mathbf{Y}_n^{(2)} = \Lambda\, \eta_n^{(2)} + \varepsilon_n^{(2)}, \quad \Sigma^{(2)} = \Lambda \Phi^{(2)} \Lambda' + \Psi^{(2)}.
\end{aligned}
$$

In this two group model the matrix Λ is invariant over groups, but all other matrices ($\Sigma^{(g)}$, $\Phi^{(g)}$, and $\Psi^{(g)}$) are allowed to vary over groups (as denoted by the superscripts 1 or 2).

This basic model follows the general statements of the oblique version of the Confactor model made by Cattell and Cattell (1955). In these treatments we first obtain initial orthogonal solutions and then rotate these solutions to invariance. In our notation we write

$$(6) \qquad \begin{aligned} \boldsymbol{\Sigma}^{(1)} &= \boldsymbol{\Lambda}^{(1)}\boldsymbol{\Lambda}^{(1)\prime} + \boldsymbol{\Psi}^{(1)} \text{ and} \\ \boldsymbol{\Sigma}^{(2)} &= \boldsymbol{\Lambda}^{(2)}\boldsymbol{\Lambda}^{(2)\prime} + \boldsymbol{\Psi}^{(2)}, \end{aligned}$$

where $\boldsymbol{\Lambda}^{(1)}$and $\boldsymbol{\Lambda}^{(2)}$ are the $(M \times K)$ orthogonal loading matrices for each group obtained separately using some form of K^2 restrictions. Under the assumptions of Parallel Proportional Profiles, we assume the new set of expectations

$$(7) \qquad \begin{aligned} \boldsymbol{\Sigma}^{(1)} &= \boldsymbol{\Lambda}\boldsymbol{\Lambda}' + \boldsymbol{\Psi}^{(1)}, \text{ and} \\ \boldsymbol{\Sigma}^{(2)} &= \boldsymbol{\Lambda}\boldsymbol{\Delta}^2\boldsymbol{\Lambda}' + \boldsymbol{\Psi}^{(2)}, \end{aligned}$$

where $\boldsymbol{\Lambda}$ is the invariant loading matrix and $\boldsymbol{\Delta}$ is the $(K \times K)$ diagonal matrix of the K-factor standard deviations in the second group. These factor standard deviations $\boldsymbol{\Delta}$ allow the loadings $\boldsymbol{\Lambda}$ to be invariant but the common factors to be *proportional* over groups (by variance terms $\boldsymbol{\Delta}^2$).

Following Cattell and Cattell (1955) we can rewrite this model in terms of the transformation matrices applied to Equation 6 needed to obtain the invariant loadings of Equation 7: We write

$$(8) \qquad \begin{aligned} \boldsymbol{\Lambda} &= \boldsymbol{\Lambda}^{(1)}\mathbf{T}^{(1)}, \text{ and} \\ \boldsymbol{\Lambda} &= \boldsymbol{\Lambda}^{(2)}\,\mathbf{T}^{(2)}\,\boldsymbol{\Delta}^{-1}, \text{ so} \\ \boldsymbol{\Lambda}^{(1)}\mathbf{T}^{(1)} &= \boldsymbol{\Lambda}^{(2)}\,\mathbf{T}^{(2)}\,\boldsymbol{\Delta}^{-1}. \end{aligned}$$

where $\mathbf{T}^{(1)}$and $\mathbf{T}^{(2)}$ are $(K \times K)$ *orthonormal* transformation matrices (i.e., $\mathbf{T}\,\mathbf{T}' = \mathbf{T}'\,\mathbf{T} = \mathbf{I}$). Now by rearrangement of $\mathbf{T}^{(1)}$and $\boldsymbol{\Lambda}^{(2)}$ we have,

$$(9) \qquad \begin{aligned} \boldsymbol{\Lambda}^{(1)}(\mathbf{T}^{(1)}\mathbf{T}^{(1)\prime}) &= \boldsymbol{\Lambda}^{(2)}\,\mathbf{T}^{(2)}\,\boldsymbol{\Delta}^{-1}\,\mathbf{T}^{(1)\prime}, \\ \boldsymbol{\Lambda}^{(1)} &= \boldsymbol{\Lambda}^{(2)}\,\mathbf{T}^{(2)}\,\boldsymbol{\Delta}^{-1}\,\mathbf{T}^{(1)\prime}, \\ \boldsymbol{\Lambda}^{(2)\prime}\,\boldsymbol{\Lambda}^{(1)} &= (\boldsymbol{\Lambda}^{(2)\prime}\,\boldsymbol{\Lambda}^{(2)})\,\mathbf{T}^{(2)}\,\boldsymbol{\Delta}^{-1}\,\mathbf{T}^{(1)\prime}, \\ (\,\boldsymbol{\Lambda}^{(2)\prime}\,\boldsymbol{\Lambda}^{(2)}\,)^{-1}\,\boldsymbol{\Lambda}^{(2)\prime}\,\boldsymbol{\Lambda}^{(1)} &= \mathbf{T}^{(2)}\,\boldsymbol{\Delta}^{-1}\mathbf{T}^{(1)\prime}, \text{ or} \\ \boldsymbol{\Gamma} &= \mathbf{T}^{(2)}\,\boldsymbol{\Delta}^{-1}\,\mathbf{T}^{(1)\prime}, \text{ or} \\ \boldsymbol{\Gamma} &= \mathbf{U}\gamma\,\mathbf{S}\gamma\,\mathbf{V}\gamma', \end{aligned}$$

where $\boldsymbol{\Gamma}$ is the $(K \times K)$ square but non-symmetric matrix formed from the known $\boldsymbol{\Lambda}^{(1)}$and $\boldsymbol{\Lambda}^{(2)}$ orthogonal matrices.

The *basic structure* or *singular value decomposition* of this $\mathbf{\Gamma}$ matrix (see Horst, 1963; Johnson, 1963; Eckhart & Young, 1936) gives a unique solution for the unknown matrices: (a) $\mathbf{T}^{(2)} = \mathbf{U}\gamma$, the left orthonormal vectors of $\mathbf{\Gamma}$, (b) $\mathbf{T}^{(1)} = \mathbf{V}\gamma$, (b) the right orthonormal vectors of $\mathbf{\Gamma}$, and (c) $\mathbf{\Delta} = \mathbf{S}\gamma^{-1}$, the recripricols of the singular values of $\mathbf{\Gamma}$. A version of this proof was implied by Cattell and Cattell (1955) and given by Meredith (1964a).

If the two-group orthogonal Confactor model Equation 7 is appropriate then the resulting $\mathbf{\Lambda}$ can be rotated to an identical position across groups. Under these conditions we can obtain parameter estimates for *all* ($M \times K$) loadings in the $\mathbf{\Lambda}$ matrix even though this $\mathbf{\Lambda}$ does not necessarily have a simple structure form. The inital extraction of each orthogonal $\mathbf{\Lambda}^{(g)}$ required K^2 constraints but did not require invariance. Under the additional constraints of MK invariant loadings in $\mathbf{\Lambda}$, *the multiple-group Confactor model requires only* K^2 total identification constraints.

Rotational Indeterminancy in the Two-Group Model

There are several sources of indeterminacy in this two-group solution. In the simplest case we can reverse the ordering of the groups, so $\mathbf{\Phi}^{(1)}$ $\mathbf{\Delta}^{(1)}$and $\mathbf{\Phi}^{(2)} = \mathbf{I}$, and the resulting $\mathbf{\Lambda}$ will also be invariant and unique but appear different than before. This is a trivial concern because the rotated loadings $\mathbf{\Lambda}$ are simply rescaled columnwise by the inverse of the previous $\mathbf{\Delta}^{(2)}$. These rescalings should have no effect on the substantive interpretations of the model.

A second source of indeterminacy is far more complex. We know the invariant $\mathbf{\Lambda}$ obtained above is unique only under the orthogonal factor assumptions. However, if we allow the factors in, say, the second group to be freely correlated ($\mathbf{\Phi}^{(2)}$), this relaxes $K(K - 1)/2$ of the needed constraints and we cannot uniquely estimate all ($M \times K$) parameters in the rows and columns of $\mathbf{\Lambda}$. Following the one-group orthogonal case above we can identify the model by adding the needed $K(K - 1)/2$ restrictions within the loading matrix $\mathbf{\Lambda}$. These restrictions are usually in the form of "fixed zeros" and these usually reflect a substantive hypotheses (McDonald, 1985).

Similar problems surface in the general oblique case. The initial two-group orthogonal calculation required the orthonormal restrictions ($\mathbf{TT}' = \mathbf{T}'\mathbf{T} = \mathbf{I}$). It follows that the resulting $\mathbf{\Lambda}$ can be rotated using non-orthogonal transformations. As before, we can start with an initial invariant $\mathbf{\Lambda}_i$ identified by assuming $\mathbf{\Phi}_i^{(1)} = \mathbf{I}$, and $\mathbf{\Phi}_i^{(2)} = \mathbf{\Delta}_i^2$. Now we can select a transfomation matrix

(10)
$$\mathbf{T} = \mathbf{\Lambda}_s^{-1}, \text{ so } \mathbf{\Lambda} = \begin{bmatrix} \mathbf{I} \\ \mathbf{\Lambda}_u \mathbf{\Lambda}_s^{-1} \end{bmatrix}, \text{ and}$$
$$\mathbf{\Phi}^{(1)} = \mathbf{\Lambda}_s \mathbf{\Lambda}_s' = \mathbf{\Delta}^{(1)}\mathbf{R}^{(1)}\mathbf{\Delta}^{(1)},$$
$$\mathbf{\Phi}^{(2)} = \mathbf{\Lambda}_s \mathbf{\Delta}_i^2 \mathbf{\Lambda}_s' = \mathbf{\Delta}^{(2)} \mathbf{R}^{(2)} \mathbf{\Delta}^{(2)},$$

so each group has a new invariant $\mathbf{\Lambda}$ but possibly different factor covariance matrices $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$. These factor covariances are also written here with different factor standard deviations $\mathbf{\Delta}^{(1)}$ and $\mathbf{\Delta}^{(2)}$ and different factor correlations $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$. In this development $\mathbf{R}^{(1)} \neq \mathbf{R}^{(2)}$ because, in general, $\mathbf{\Delta}_i \mathbf{\Lambda}_s \neq \mathbf{\Lambda}_s \mathbf{\Delta}_i$ unless further restrictions are imposed. The oblique rotation of an invariant $\mathbf{\Lambda}$ does not necessarily imply equal factor correlation matrices (as in Harshman & Lundy, 1984; cf. Meredith, 1964a; Thurstone, 1947).

We have just described different sets of alternative constraints that yield an invariant $\mathbf{\Lambda}$. However, if these alternatives yield identical fit to the data, we can show they all have the same basic structure. (a) Let $\mathbf{\Lambda}_d$ be the invariant loading matrix obtained by requiring both $\mathbf{\Phi}^{(1)} = \mathbf{I}$ and $\mathbf{\Phi}^{(2)} = \mathbf{\Delta}^2$ to be *diagonal* but allowing all *MK* factor loading parameters to be free. (b) Let $\mathbf{\Lambda}_t$ be the invariant loading matrix obtained by requiring $\mathbf{\Phi}^{(1)} = \mathbf{I}$ and requiring $K(K - 1)/2$ lower triangular factor loading parameters to be fixed. (c) Let $\mathbf{\Lambda}_r$ be the invariant loading matrix obtained by allowing both $\mathbf{\Phi}^{(g)}$ to be free and requiring K^2 reference variable loading parameters to be fixed. Following an approach suggested by Meredith (personal communication, November, 1988) we now write the basic structure decomposition of these common factor matrices as

(11)
$$\begin{aligned} \mathbf{\Lambda}_d \mathbf{\Lambda}_d' &= (\mathbf{U}_d \mathbf{S}_d \mathbf{V}_d')(\mathbf{V}_d' \mathbf{S}_d \mathbf{U}_d)' &&= \mathbf{U}_d \mathbf{S}_d^2 \mathbf{U}_d', \\ \mathbf{\Lambda}_t \mathbf{\Lambda}_t' &= (\mathbf{U}_t \mathbf{S}_t \mathbf{V}_t')(\mathbf{V}_t' \mathbf{S}_t \mathbf{U}_t)' &&= \mathbf{U}_t \mathbf{S}_t^2 \mathbf{U}_t', \text{ and} \\ \mathbf{\Lambda}_r \mathbf{\Phi}^{(1)}\mathbf{\Lambda}_r' &= (\mathbf{U}_r \mathbf{S}_r \mathbf{V}_r') \mathbf{\Phi}^{(1)}\mathbf{V}_r' \mathbf{S}_r \mathbf{U}_r)' &&= \mathbf{U}_r (\mathbf{U}_p \mathbf{S}_{r\phi}^2 \mathbf{U}_p') \mathbf{U}_r' \\ & &&= \mathbf{U}_{r\phi} \mathbf{S}_{r\phi}^2 \mathbf{U}_{r\phi}', \end{aligned}$$

with orthonormal vectors $\mathbf{U}_x \mathbf{U}_x' = \mathbf{I}$, $\mathbf{V}_x \mathbf{V}_x' = \mathbf{I}$, and values $\mathbf{S}_x$. Assuming each of these alternative models produces the same $\mathbf{\Sigma}^{(1)}$ and $\mathbf{\Psi}^{(1)}$ we can write

(12)
$$\begin{aligned} \mathbf{\Sigma}^{(1)} &= \mathbf{\Lambda}_d \mathbf{\Lambda}_d' + \mathbf{\Psi}^{(1)}, \\ \mathbf{\Sigma}^{(1)} &= \mathbf{\Lambda}_t \mathbf{\Lambda}_t' + \mathbf{\Psi}^{(1)}, \text{ and} \\ \mathbf{\Sigma}^{(1)} &= \mathbf{\Lambda}_r \mathbf{\Phi}^{(1)}\mathbf{\Lambda}_r' + \mathbf{\Psi}^{(1)}, \text{ so} \end{aligned}$$

$$\begin{aligned} \mathbf{\Lambda}_d \mathbf{\Lambda}_d' &= \mathbf{\Lambda}_t \mathbf{\Lambda}_t' &&= \mathbf{\Lambda}_r \mathbf{\Phi}^{(1)}\mathbf{\Lambda}_r', \text{ or} \\ \mathbf{U}_d \mathbf{S}_d^2 \mathbf{U}_d' &= \mathbf{U}_t \mathbf{S}_t^2 \mathbf{U}_t' &&= \mathbf{U}_{r\phi} \mathbf{S}_{r\phi}^2 \mathbf{U}_{r\phi}', \text{ so} \\ \mathbf{U}_d \mathbf{S}_d &= \mathbf{U}_t \mathbf{S}_t &&= \mathbf{U}_{r\phi} \mathbf{S}_{r\phi}. \end{aligned}$$

Thus, the alternative but invariant solutions Λ_d, Λ_t, or Λ_r all have an orthogonal basic structure defined by $\mathbf{U}_d$, $\mathbf{U}_t$, or $\mathbf{U}_{r\phi}$. Since any orthogonal rotation can be defined as a product of orthogonal transformations (Horst, 1963), these $\mathbf{U}_x$ matrices can be rotated to the same position by an appropriate choice of a non-singular transformation matrix. (The equivalent representation can be written for $\Sigma^{(2)}$.) Thus, although the initial choice of identification constraints across groups is usually arbitrary (as in Λ_d, Λ_t, or Λ_r, above) *the invariant* Λ *matrix retains the same basic structure information under any choice of initial constraints.*

These results highlight three key issues of oblique Confactor estimation. First, given the typical restrictions of orthogonality and scaling, the $(M \times K)$ parameters of invariant Λ are unique. Second, any free parameters among the covariances of the factors $\Phi^{(g)}$ in any group requires a corresponding restriction of the invariant loadings Λ to obtain a unique solution. Third, there are many choices for the placement and size of the required fixed values in Λ, so, in the oblique solution, the initial Λ_i may be rotated. That is, the covariance prediction of $\Sigma^{(g)}$ remains identical under any choice of the placement of K^2 fixed values in Λ and $\Phi^{(g)}$. The gain in parsimony with multiple groups may now be clear: When dealing with two separate loadings matrices $\Lambda^{(g)}$ we need K^2 constraints in each group to define a unique rotation. But now, by including the restrictions of invariance in loadings Λ over groups, we can define a unique rotation with only K^2 total constraints.

Factorial Invariance in More than Two Groups

Now we can extend these basic principles to include more than two groups. Let us assume we have measured the same M variables on G groups with possible different sample sizes ($N^{(g)}$, for g = 1 to G). We write a K-factor model in each group as

$$\text{(13)} \qquad \begin{aligned} \mathbf{Y}_n^{(g)} &= \Lambda\, \eta_n^{(g)} + \varepsilon_n^{(g)}, \\ \Sigma^{(g)} &= \Lambda\, \Phi^{(g)}\, \Lambda' + \Psi^{(g)}, \end{aligned}$$

where Λ is invariant across all G groups, but all other matrices are allowed to vary for the gth group. As before, we can start with an initial solutions in all groups with $\Lambda_i^{(g)}$ where K^2 restrictions are implied in each group but invariance is not initially required. In one extension of the Confactor model Equation 7 we can add factor covariance restrictions for the G groups as

$$(14) \qquad \begin{aligned} \mathbf{\Phi}^{(1)} &= \mathbf{I}, \\ \mathbf{\Phi}^{(2)} &= \mathbf{\Delta}^{(2)}\,\mathbf{\Delta}^{(2)}, \text{ and} \\ \mathbf{\Phi}^{(g)} &= \mathbf{\Delta}^{(g)}\,\mathbf{R}^{(g)}\,\mathbf{\Delta}^{(g)}, \end{aligned}$$

so the first matrix is orthogonal, the second matrix is diagonal, and the other G - 2 matrices are generally free to vary (i.e., $\mathbf{R}^{(g)}$ is an unrestricted correlation matrix). This extension of Confactor model Equations 7 and 8 can be written as

$$(15) \qquad \begin{aligned} \mathbf{\Lambda} &= \mathbf{\Lambda}_i^{(1)}\mathbf{T}^{(1)}, \\ \mathbf{\Lambda} &= \mathbf{\Lambda}_i^{(2)}\,\mathbf{T}^{(2)}\,\mathbf{\Delta}^{(2)-1} \text{ and} \\ \mathbf{\Lambda} &= \mathbf{\Lambda}_i^{(g)}\,\mathbf{T}^{(g)}\,\mathbf{U}_\phi^{(g)}\,\mathbf{S}_\phi^{(g)-1}, \text{ so} \\ \mathbf{\Lambda}_i^{(1)}\mathbf{T}^{(1)} &= \mathbf{\Lambda}_i^{(2)}\,\mathbf{T}^{(2)}\,\mathbf{\Delta}^{(2)-1} = \mathbf{\Lambda}_i^{(g)}\,\mathbf{T}^{(g)}\,\mathbf{U}_\phi^{(g)}\,\mathbf{S}_\phi^{(g)-1}, \end{aligned}$$

where all $\mathbf{T}^{(g)}$ matrices are orthonormal, and $\mathbf{S}_\phi^{(g)}$ and $\mathbf{U}_\phi^{(g)}$ are the singular values and vectors of $\mathbf{\Phi}^{(g)}$ (for $g = 3$ to G).

The previous statement is useful because it suggests a simple calculation technique. First, we obtain an invariant $\mathbf{\Lambda}$ from the previous two-group expression, Equation 9, using either the resulting $\mathbf{T}^{(1)}$or $\mathbf{T}^{(2)}$. Second we define the non-orthogonal but nonsingular transformation matrix $\mathbf{T}_\phi^{(g)} = \mathbf{T}^{(g)}\,\mathbf{U}_\phi^{(g)}\,\mathbf{S}_\phi^{(g)}$ and write

$$(16) \qquad \begin{aligned} \mathbf{\Lambda}^{(g)}\,\mathbf{T}_\phi^{(g)} &= \mathbf{\Lambda}, \\ (\mathbf{\Lambda}^{(g)\prime}\,\mathbf{\Lambda}^{(g)})\,\mathbf{T}_\phi^{(g)} &= \mathbf{\Lambda}^{(g)\prime}\,\mathbf{\Lambda}, \text{ so} \\ \mathbf{T}_\phi^{(g)} &= (\mathbf{\Lambda}^{(g)\prime}\,\mathbf{\Lambda}^{(g)})^{-1}\,\mathbf{\Lambda}^{(g)\prime}\,\mathbf{\Lambda}, \text{ and} \\ \mathbf{\Phi}^{(g)} &= \mathbf{T}_\phi^{(g)-1}\,\mathbf{\Phi}_i^{(g)}\,\mathbf{T}_\phi^{(g)-1\prime} = \mathbf{\Delta}^{(g)}\,\mathbf{R}^{(g)}\,\mathbf{\Delta}^{(g)}. \end{aligned}$$

A more formal proof of this assertion can be developed from an extension of this result using theorems provided by Anderson (1958, p.341, Theorem 3; as suggested by Browne, personal communication, September, 1988).

The previous Equation 16 results in a matrix $\mathbf{\Lambda}$ which is invariant over all groups, but the underlying common factors are (a) orthogonal with unit variance in group 1, (b) orthogonal with different variances in group 2, and (c) oblique with additionally different variances in all other groups ($g = 3$ to G). Since this $\mathbf{\Lambda}$ was obtained from the first two groups and simply applied to the other G - 2 groups, it is clear that only the minimum of K^2 restrictions are required. In this approach, the covariance information $\mathbf{\Sigma}^{(g)}$ from the other G - 2 groups contributes to the estimation of $\mathbf{\Lambda}$ but all factor loadings can be fixed by the identification of groups one and two alone.

All rotations demonstrated earlier apply to this unique solution. First, as in the two-group case above, this $\mathbf{\Lambda}$ can be rescaled column-wise by altering the group constraints. Second, and more critically, any invariant $\mathbf{\Lambda}$ can be rotated by altering the restrictions on the group covariances. We can start with some initial solutions for $\mathbf{\Lambda}_i$, identified by restricting $\mathbf{\Phi}^{(1)} = \mathbf{I}$, $\mathbf{\Phi}^{(2)} = \mathbf{\Delta}_i^2$, and then write

$$(17) \qquad \mathbf{T} = \mathbf{\Lambda}_s^{-1}, \qquad \mathbf{\Lambda} = \begin{bmatrix} \mathbf{I} \\ \mathbf{\Lambda}_u \mathbf{\Lambda}_s^{-1} \end{bmatrix}, \text{ and}$$
$$\mathbf{\Phi}^{(1)} = \mathbf{\Lambda}_s \mathbf{\Lambda}_s' = \mathbf{\Delta}^{(1)}\mathbf{R}^{(1)}\mathbf{\Delta}^{(1)},$$
$$\mathbf{\Phi}^{(2)} = \mathbf{\Lambda}_s \mathbf{\Delta}_i^2 \mathbf{\Lambda}_s' = \mathbf{\Delta}^{(2)} \mathbf{R}^{(2)} \mathbf{\Delta}^{(2)},$$
$$\mathbf{\Phi}^{(g)} = \mathbf{\Lambda}_s \mathbf{\Phi}_i^{(g)} \mathbf{\Lambda}_s' = \mathbf{\Lambda}_s \mathbf{\Delta}_i^{(g)} \mathbf{R}_i^{(g)} \mathbf{\Delta}_i^{(g)} \mathbf{\Lambda}_s' = \mathbf{\Delta}^{(g)} \mathbf{R}^{(g)} \mathbf{\Delta}^{(g)},$$

so now the covariances $\mathbf{\Phi}^{(g)}$, deviations $\mathbf{\Delta}^{(g)}$, and correlations $\mathbf{R}^{(g)}$, are free to vary among the G groups. This $\mathbf{\Lambda}$ will be unique under the restrictions above but it can be rotated in a variety of ways. As before, the placement of these K^2 restrictions alters the specific values of the invariant $\mathbf{\Lambda}$. However, as before, the basic orthogonal decomposition of any $\mathbf{\Lambda}$ into vectors $\mathbf{U}_x$ and values $\mathbf{S}_x$ will provide the same information from any arbitrary set of restrictions.

Comparing the Required Number of Parameters

We have shown that if we require a minimum of only K^2 constraints for any number of groups G then the corresponding invariant $\mathbf{\Lambda}$ can have $(M \times K)$ free parameters. These general principles become more useful when more groups are added because no new restrictions are required. One practical benefit becomes clear when we compare the number of independent parameters required in the fitting of alternative models (see Figure 1 and 2).

The number of parameters (*Npar*) in any model can be defined as the sum of the free parameters (loadings, covariances, and uniquenesses) minus the required constraints. First we write the number of parameters required for the independent and exactly-identifed solution in each group as

$$(18) \qquad Npar_{\text{independent}} = G\,(M\,K + [K(K+1)/2] + M - K^2)$$
$$= G\,K\,(M - K) + G\,[K(K+1)/2 + M].$$

Alternatively, the model of Figure 1 is a Configural simple structure model where the *m*th variable has only one loading on the *k*th factor, and these loadings are not invariant across the G groups. This is a restriction of the previous independent groups model and we write

$$(19) \qquad Npar_{\text{configural}} = G\,(M + [K(K+1)/2] + M - K) = G\,(M - K) + G\,[K(K+1)/2 + M].$$

Finally, the model of Figure 2 is a Confactor model where the mth variable has several loading on the K factors, and these loadings are invariant across the G groups. This is also a restriction of the previous independent groups model and now we write

$$(20) \qquad Npar_{\text{confactor}} = (K\,M) + G\,([K(K+1)/2] + M) - K^2 = K\,(M - K) + G\,[K(K+1)/2 + M].$$

The previous models can now be seen to have common parameters (i.e., $G\,[K(K+1)/2 + M]$), so the differences in the number of parameters required can be writen for each pair as

$$(21) \qquad \begin{aligned} Npar_{\text{independent}} - Npar_{\text{configural}} &= G\,(K - 1)\,(M - K), \\ Npar_{\text{independent}} - Npar_{\text{confactor}} &= (G - 1)\,(M - K), \text{ and} \\ Npar_{\text{configural}} - Npar_{\text{confactor}} &= (G - K)\,(M - K). \end{aligned}$$

These differences in the number of parameters between the models are directly based on the number of groups G and the number of factors K (and multiplied by $M - K$). The Configural model (Figure 1) is a simplification of the exactly-identified independent groups model, and this simplification increases as the number of groups G and number of factors K increases. The Confactor model (Figure 2) is a simplification of the exactly-identified independent groups model, and this simplification increases mainly as the number of groups G increases. Most critically, the Confactor model is simpler than the Configural model in cases where the number of groups G is large and the number of factors K is small (i.e., $G - K$). In contrast, the Configural model will require less parameters when the number of factors K is large and the number of groups G is small. All of these differences in the number of parameters will be enhanced with an increasing number of variables M relative to number of factors K.

Fitting the Multiple-Group Confactor Model

In structural modeling terms we view the Confactor model as a hypothesis to be fitted. In the two-group orthogonal case we can rewrite the model as

$$\begin{aligned} \text{(22)} \qquad \mathbf{\Lambda}^{(1)}\mathbf{T}^{(1)} &= \mathbf{\Lambda}^{(2)}\,\mathbf{T}^{(2)}\,\mathbf{\Delta}^{-1} + \mathbf{Q}, \text{ or} \\ \mathbf{Q} &= \mathbf{\Lambda}^{(1)}\mathbf{T}^{(1)}\,\mathbf{\Lambda}^{(2)}\,\mathbf{T}^{(2)}\,\mathbf{\Delta}^{-1}, \end{aligned}$$

where **Q** is a $(M \times K)$ matrix of differences between the theoretical and the observed loadings after rotation. In the case of more groups we define **Q** to represent the differences between all loading matrix expectations. To fit this model we select an appropriate index of fit (e.g., $tr[\mathbf{QQ}']$), search for a specific rotation which minimizes this discrepancy or misfit, and obtain a solution such as Equation 9 above. Meredith (1964b) details techniques for defining rotations which optimize factorial invariance (also see Gow, 1978).

As an alternative we can fit the two-group Confactor model Equation 7 using standard structural modeling algorithms (e.g., LISREL). In this structural modeling approach we assume $\mathbf{\Lambda}$ invariance from the start and evaluate the goodness-of-fit from a unique rotational position. We initially require all model to be identified and, as we have just demonstrated, this can be achieved by requiring: (a) orthogonality in a first group ($\mathbf{\Phi}^{(1)} = \mathbf{I}$), (b) diagonality in a second group ($\mathbf{\Phi}^{(2)} = \mathbf{\Delta}^2$), and (c) invariance of all $(M \times K)$ factor loadings in $\mathbf{\Lambda}$. But here we do not calculate the initial solutions $\mathbf{\Lambda}^{(g)}$ and then rotate these solutions to invariance. Instead we directly estimate parameters (and standard errors) for *all* $(M \times K)$ loadings in the invariant $\mathbf{\Lambda}$ matrix.

In this structural modeling approach we write a fitting function

$$\begin{aligned} \text{(23)} \qquad F &= \sum_{g=1}^{G} [(N^{(g)}/N)F^{(g)}], \text{ and} \\ F^{(g)} &= \tfrac{1}{2}\, tr\{[(\mathbf{S}^{(g)} - \mathbf{\Sigma}^{(g)})\mathbf{W}^{(g)-1}]^2\}, \end{aligned}$$

where, for g groups, $\mathbf{S}^{(g)}$ is the observed covariance matrix, $\mathbf{W}^{(g)}$ is the weight matrix, $F^{(g)}$ is the function or discrepancy value, and F is the function value over all groups. Given the appropriate choice of weights $\mathbf{W}^{(g)}$ we obtain the usual least squares ($\mathbf{W}^{(g)} = \mathbf{I}^{(g)}$), weighted least squares ($\mathbf{W}^{(g)} = \mathbf{S}^{(g)}$), or approximate maximum likelihood ($\mathbf{W}^{(g)} = \mathbf{\Sigma}^{(g)}$) fitting functions (for details, see Chen & Robinson, 1985; Jöreskog & Sörbom, 1979, 1985; McDonald, 1985). In all applications here we will estimate parameters to mimimize a maximum likelihood fitting function, and we will estimate a likelihood ratio test statistic χ^2 to evaluate the goodness-of-fit. In this approach, the resulting $\mathbf{\Lambda}$ will always be invariant, but the oblique Confactor model may not fit the data very well. These results will be demonstrated in the applications of the next section.

In using structural models we also examine this identification problem on a numerical basis (see McDonald & Krane, 1979; Shapiro & Browne, 1983). Specifically, when a numerical search procedure no longer finds parameters for a smaller fitting function, we can check the rank of the inverse of the matrix of second order derivatives at the minimum (i.e., the Fisher Information matrix). If this matrix is of full rank the model can be said to be identified. The numerical properties of this matrix are also reflected in the main diagonal elements of this matrix, and these are given as the standard errors for each parameter. The numerically determined identification status of these newer models will be examined in all results to follow.

Practical Structural Modeling Solutions

The previous results lead to some practial approaches to Confactor solutions using structural equation techniques. We have shown how to obtain a final solution for the Confactor model if we are certain about our selection of the two orthogonal reference groups 1 and 2. Unfortunately, this selection of reference groups is usually arbitrary and the factor loadings Λ_i can be rotated into a large set of other matrices Λ and covariances $\Phi^{(g)}$ all of which produce the same communalities and goodness-of-fit to the data. This rotation problem was not explicitly stated by Cattell (1944; Cattell & Cattell, 1955; Cattell & Brennan, 1977). In earlier work Cattell seemed to believe that there was only one position where the factors could be invariant and proportional over groups. McArdle and Cattell (1988) used the equations here to show that a unique position can be found using structural equation models, but also that alternative selections of the required K^2 constraints leads to alternative rotations of the invariant common factor space.

But the benefits of this multiple group Confactor solution have not been fully explored. In the Confactor model the number and proportion of oblique covariances $\Phi^{(g)}$ uniquely estimated increases as a simple function of the number of groups G. However, in contrast to G separate group solutions with $\Lambda^{(g)}$ separate matrices, we benefit from having only one Λ matrix to rotate. Three different structural modeling solutions for this purpose are described next.

Solution 1: Selecting a Good Reference Variable Rotation

The original approach of Cattell (1944) suggested an avoidance or minimal use of the principle of simple structure. If we admit these solutions, however, we might now try to find an optimal position for the minimal number of zero locations in the factor loading pattern $\mathbf{\Lambda}$. The selection of a unique reference variable submatrix $\mathbf{\Lambda}_s$ from $\mathbf{\Lambda}$ can only be accomplished in a finite number of ways. This number can be written as a binomial coefficient

$$(24) \qquad \left(\frac{M}{K} \right) = \left(\frac{M!}{K!} \right) (M - K)!$$

representing the number of ways of choosing K rows from a set of M rows without regard to order. In many cases, it will be possible to explore all of these separate rotations (e.g., with $M = 9$ and $K = 3$, so $9!/3!(9 - 3)! = 84$).

The transformation matrix $\mathbf{T} = \mathbf{\Lambda}_s^{-1}$ needs to be recalculated for each set of K variables, but the information in this calculation is useful. The quality of the new loadings estimates ($\mathbf{\Lambda}_u \mathbf{\Lambda}_s^{-1}$) should be indicated by the numerical characteristics of the inverse; e.g., as the selected variables tend towards being uncorrelated the inverse matrix will approach diagonality. Some useful features of a final reference variable solution can be indexed by some function of the determinant of $\mathbf{\Lambda}_s$ (and we use $ln|\mathbf{\Lambda}_s|$). Transformation matrices with the largest determinants should index variables at the extremes of the conic sphere formed by the vector projections. Thus, we may be able to obtain an optimal set of K reference variables by the empirical rotation of all M/K different transformation matrices defined by Equation 24. The empirical selection of a good reference variable set will be checked empirically in the next section.

Solution 2: Basic Structure Rotation

A more general exploratory approach to rotation is possible. Since the goodness-of-fit of the exactly identified invariant $\mathbf{\Lambda}$ is always the same, we know that model is invariant across restrictions. Following Meredith (1964b, 1965) we could create a pooled factor covariance matrix $\bar{\mathbf{\Phi}}$ pre and post multiplied by $\mathbf{\Lambda}$ and followed by a singular value decomposition. This approach has conceptual appeal but it is not strictly required. As Meredith (personal communication, November, 1988) has recently pointed out, we can also accomplish this goal by applying the singular value decomposition directly to any invariant $\mathbf{\Lambda}$ and rotate the orthonormal vectors $\mathbf{U}_\lambda$ by Varimax, Promax, Oblimin, Hyball, Procrustes, or any other rotation scheme.

In a first step we estimate a unique and invariant factor pattern $\mathbf{\Lambda}$ by using a structural equation program for multiple groups (e.g., LISREL; see Appendix 2). Second, we calculate the singular value decomposition of this invariant Λ using an external routines. Third, we apply some external rotation of the invariant U_λ using any criteria defined above. Fourth, we create a new set of K^2 constraints from the the specific rotational values, refit the structural equation model, and obtain new parameter estimates and standard errors for all free parameters. This kind of multi-stage solution should be be practical, efficient, and accurate.

We should note that simultaneous solutions combining invariance and rotational criteria have been suggested (Bloxom, 1972) and are rapidly becoming practical (see Browne & Du Toit, 1987). In such an approach we could, for example, estimate the invariant Confactor model and also minimize, say, the Varimax criterion for each factor from the estimated loadings (see Horst, 1963). This nonstandard approach allows (a) an oblique invariant model with any simplicity criterion applied to the final pattern, (b) any degree of over-identification by restrictive hypotheses, and (c) the direct calculation of standard errors for the simultaneous solution.

Despite these potential benefits the simultaneous solution is not usually needed here. In the oblique Confactor model proposed here we require invariance first and some additional rotational criteria second. Since the solution is exactly-identified we should obtain a similar solution by either simultaneous criteria or by a two step criteria (within the limits of rounding error). In any case, the simultaneous solution Λ would be rotatable into any of the other positions. On these practical and theoretical grounds we now use the multi-stage analysis to demonstrate some numerical results.

Solution 3: Adding Structural Constraints

We have just described how additional restrictions on the factor covariances $\Phi^{(g)}$ lead to less restrictions required on the factor loadings Λ. Now we ask the question, "What meaningful mathematical and statistical restrictions can be placed on the factors over groups?" Cattell (1966) suggested the inclusion of patterning of higher order factors and group means, and we explore these possibilities now.

In a first kind of model we will restrict the covariances among the factors to be patterned with an invariant higher order factor (i.e., $\Phi^{(g)} = \mathbf{B}\,\Omega^{(g)}\,\mathbf{B}' + \gamma^{(g)}$). This will allow all factors in all groups to be freely correlated, but it will not require a full set of restrictions on the invariant Λ. We can also explore the possibility of similar restrictions based on latent path models. Since any fully recursive path model can be written as a set of higher order models (as in McArdle & McDonald, 1984; McDonald, 1980) these relationships should be similar. There are several restrictive latent path models based on substantive concerns which may be used to identify the factor loadings (as in McArdle, 1984b).

The inclusion of latent variable means is a critical aspect of recent structural modeling (see Sörbom, 1974; Horn & McArdle, 1980, 1992; McArdle, 1988; Millsap & Everson, 1991). In models where the factor pattern Λ is invariant over groups it is possible to examine the mean differences between groups on these same factors (i.e., $\mu^{(g)} = \upsilon + \Lambda\, \Theta^{(g)}$). Furthermore, the inclusion of factors that represent covariance differences and mean differences adds another important aspect to the determination of an invariant factor pattern. Using these techniques we can define factors that simultaneously account for both differences within persons and between groups of persons (after Cattell, 1966; Meredith, 1990; Horn & McArdle, 1992). Additional restrictions on the mean parameters (Θ) may be required to identify the appropriate loadings Λ (Meredith, 1990). The empirical identification status of these newer models will be examined in all results to follow.

Some caution needs to be expressed here because the rotation principles just described do not necessarily apply to over-identified solutions. For example, we might choose to restrict $\Phi^{(1)} = \mathbf{I}$, $\Phi^{(2)} = \Delta^{(2)}\,\Delta^{(2)}$, $\Phi^{(g)} = \Delta^{(g)}\,\Delta^{(g)}$, so the factors are all uncorrelated in all groups. This adds an additional $K(K - 1)/2$ restrictions which, although are not initially necessary, do serve to fix the rotation. Of course, any of the G - 2 matrices $\Phi^{(g)}$ can be tested for diagonality, identity, or equality, but these results are usually over-identified and are more restrictive than necessary. These kinds of structural restrictions may or may not be beneficial.

Results of Structural Confactor Analysis

Multiple Groups and Multiple Measures Data

To examine some new structural solutions we use the well-known data selected by Meredith (1964a, 1964b).

> The data used to provide a numerical illustration of the techniques presented here are taken from a monograph by Holzinger and Swineford [1939]. In this study 25 carefully select tests were administered to seventh and eighth grader students in two schools. The socioeconomic character of the two schools was quite different, one (Pasteur) enrolling children of factory workers, a large percentage of whom were foreign born, and the other (Grant-White) enrolling children in a middle class suburban area. ... In addition to the results of various analyses the monograph contains complete data for all 301 children tested, including

> all test scores and factor scores, making it possible to divide the students in each school into two approximately equal groups by splitting at the median, within each schools on one of the speeded tests, an addition test. This yielded four groups for the purposes of the analyses to be presented here. ... Nine of the 25 possible tests were chosen for the purpose of illustration. The tests were chosen so that the space, verbal, and memory factors are each represented by three tests. ... The first six tests are all multiple choice. In the Figure Recognition test the subjects were allowed to study a list of abstract figures and then indicated which figures of a larger list were members of the memorized list... (Meredith, 1964b, p.197-198).

The means, standard deviations, and correlation matrices for the four group problem are listed by Meredith (1964b, p.198; see Appendix 1 here). In the first group the means have been scaled to zero and the standard deviations have been scaled to one. The other three groups have been rescaled using the same group 1 means and deviations. This choice of the first group, or the sum over groups, is arbitrary, but the scaling of all variables using the same constants is necessary for the models to follow. All structural models below will be fitted simultaneously to the four (9 x 9) covariance matrices (including 180 overall summary statistics) and some models will include the four (9 x 1) mean vectors as well (adding 36 additional summary statistics).

These data have also been reanalyzed by Jöreskog (1971) and by Bloxom (1972) in a multiple group structural equation model (using the original SIFASP program). Also, Sörbom (1974) fitted these data in a multiple group structural equation model with means (using the COFAMM program). Comparable models will now be fit using the widely available LISREL program (also see the LISREL-7 manual, Jöreskog & Sörbom, 1985; and see Appendix 2 here). These models can also be fit by other single group programs (e.g., the COSAN program by Fraser, 1979) but the different sized samples make the multiple group LISREL generally easier to use for this example.

Result 1a: Traditional Invariance Examples

In Table 1 we list the maximum likelihood results of two models originally fitted and discussed by Jöreskog (1971, p.205). These models require a simple patterning for each of three common factors based on Meredith's (1964b) initial selection of tests. This pattern is designated on an a priori basis by the requirement of fixed zero values (labelled with "0 =" in all tables here). Table 1 includes the factor loadings $\Lambda^{(g)}$, factor standard

Table 1
Numerical Results of VERY SIMPLE STRUCTURE Multiple Group Structural Equation Models Fitted to Meredith's Four Group Covariance Matrices

[1]: $\Lambda^{(g)}$ = Factor Loadings

	Model 1 Configural Invariance			Model 2 Metric Invariance		
Variables	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$
1. Visual Perception	[.76.62.63.68]*	.0 =	.0 =	.80*	.0 =	.0 =
2. Rotating Cubes	[.54.24.40.28]*	.0 =	.0 =	.42*	.0 =	.0 =
3. Paper Form Board	[.57.39.44.58]*	.0 =	.0 =	.58*	.0 =	.0 =
4. General Information	.0 =	[.83.73.77.77]*	.0 =	.0 =	.82*	.0 =
5. Sentence Completion	.0 =	[.87.87.76.70]*	.0 =	.0 =	.85*	.0 =
6. Word Classification	.0 =	[.74.77.71.80]*	.0 =	.0 =	.80*	.0 =
7. Figure Recognition	.0 =	.0 =	[.72.42.36.54]*	.0 =	.0 =	.56*
8. Object-Number Pairs	.0 =	.0 =	[.45.63.35.51]*	.0 =	.0 =	.54*
9. Number-Figure Pairs	.0 =	.0 =	[.41.45.46.67]*	.0 =	.0 =	.59*

[2]: $\mathbf{\Delta}^{(g)}$ = Factor Standard Deviations

Group		$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$
1.	Pasteur/Low	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =
2.	Pasteur/High	1.0 =	1.0 =	1.0 =	.73*	.97*	.84*
3.	Grant/Low	1.0 =	1.0 =	1.0 =	.81*	.91*	.68*
4.	Grant/High	1.0 =	1.0 =	1.0 =	.86*	.90*	1.02*

[3]: $\mathbf{R}^{(g)}$ = Factor Correlations

Group		$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
1.	Pasteur/Low	.32*	.61*	.24	.34*	.55*	.22
2.	Pasteur/High	.60*	.09	.20	.59*	.11	.21
3.	Grant/Low	.68*	.83*	.68*	.68*	.83*	.68*
4.	Grant/High	.56*	.91*	.33*	.54*	.89*	.33*

[4]: Goodness of Fit

Parameters Estimated	$Npar_{model1} = 57$	$Npar_{model2} = 39$
Degrees of Freedom	$df_{model1} = 123$	$df_{model2} = 141$
Likelihood Ratio	$\chi^2_{model1} = 159.$	$\chi^2_{model2} = 173.$

Note. All values are Maximum Likelihood Estimates from LISREL-7 (see Appendix) from 9 variable Covariance Matrices of Four Groups (45 × 4 = 180 summary statistics). For model simplicity, variable Uniquenesses were fixed to be equal across groups. Model 1 estimated $\mathbf{\Psi}_{m,m}$ = [.43*, .55*, .70*, .33*, .25*, .45*, .46*, .76*, .67*], and Model 2 estimated $\mathbf{\Psi}_{m,m}$ = [.42*, .56*, .70*, .34*, .25*, .45*, .49*, .78*, .64*].
* Indicates a Free Parameter where $MLE_p > 2SE_p$. Labels for fixed parameters are ".0 =" is a fixed zero and "1.0 =" is a fixed unity.

deviation vectors $\Delta^{(g)}$, and factor correlations $\mathbf{R}^{(g)}$. Parameter estimates which are twice as large as their own standard errors (i.e., the parameter *t*-value > 2.0) are highlighted (with *). Goodness-of-fit indices are listed in the last rows of the table. For simplicity of presentation here we have further required the uniqueness vector Ψ to be invariant over groups.

In the first model of Table 1 we have estimated a three factor model with the same configuration of zeros and non-zeros in $\Lambda^{(g)}$. An oblique model is over-identified in each group by fixing the scale of each factor. In this model we require unit values in each Δ but allow each correlation matrix $\mathbf{R}^{(g)}$ to be freely estimated. This *Configural Invariance* patterning of Model 1 requires 57 parameters, so it has DF=123 and obtains a Likelihood Ratio χ^2=159.

In Model 2 of Table 1 we have required the factor loadings in Λ to be invariant over all four groups but we have relaxed several other constraints. The standard deviations of the factors in the first group are fixed at unit length (i.e., $\Delta^{(1)}$ are "1.0=") but this restriction allows the other three $\Delta^{(g)}$ to be freely estimated and interpreted as proportions of the first group (e.g., Factor 1 in Group 2 is .73 smaller in size than Factor 1 in Group 1). Once again, each correlation matrix $\mathbf{R}^{(g)}$ is freely estimated. This *Metric Invariance* patterning of Model 2 requires 39 parameters, so it has DF=141 and obtains a Likelihood Ratio χ^2=173.

Jöreskog (1971) used the fitting of a sequence of hypotheses to determine the appropriateness of different factor models. Model 1 has an identical configuration of factors, but these are not restricted to be identical and proportional over groups. In Model 2 the factors are both invariant and proportional over groups. These invariance constraints can alter the variable communalities so, in general, the Confactor model may not fit the data. However, Model 2 above can be represented as a special subset of Model 1 if the 36 loadings are proportional over groups (i.e., $\lambda_{m,k}^{(g)} = \lambda_{m,k}\, \Delta^{(g)}$). The difference between these models yields a difference $d\chi^2$=14 on dDF=18, and we regard this as a trivial difference, so we accept Model 2 as a more parsiminous model.

Model 2 is the special case of Parallel Proportional Profiles or Confactor Analysis because this model also include a minimal set of factor loadings. The structural modeling approach is a clear and flexible way to begin any Confactor problem. The simple structure hypothesis of Table 1 was supposedly driven by Meredith's (1964b) original hypotheses about the three salient variable indicators for each factor. In this sense the factor pattern is strongly confirmatory or restrictive, and reflects a "very simple structure" (see Revelle & Rocklin, 1979) or "cluster solution" (Jöreskog, 1971). We now turn to the modeling of a more complex but invariant factor pattern.

Reuslt 1b: Reference Variable Rotations

In Table 2 we present the numerical results of a reference variable patterning used (but not listed) by Jöreskog (1971, p.203). This model requires pairs of zero loadings on three variables (1, 4, and 7). Variable 1 loads only on Factor 1, variable 4 loads only on Factor 2, and variable 7 loads only on Factor 3. The fit of this model to all four groups is indexed by χ^2=127 on DF=129. The invariant and very simple structure of Model 2 is a subset of the parameters of the invariant and complex structure of Model 3. The difference in the goodness-of-fit is $d\chi^2$=32 on dDF=12, and this may be important. in this comparison of models, the addition of the 12 complex loadings seem to improve the goodness-of-fit of the invariant model.

An interpretation of the factors of Model 3 is difficult because some of the loadings are out of bounds. First, numerical problems for variable 8 (e.g., $\lambda_{8,1}$ = -2.21) reflects the collinearity of factors 1 and 3 ($\rho_{1,3}$ > .9 in three groups). Second, only ten of the variables have small standard errors, and these are mainly the original marker variables (see Table 1, Model 2). If only ten parameters are salient our expanded solution should not add much new information, and it probably should not fit much better. The main problem comes because this solution is exactly identified and may be rotated in a large number of ways. This problem can be demonstrated by choosing a different set of three variables as the reference variables.

In a first series of rotations we used the reference variable approach outlined earlier (see Solution 1, p. 84). A computer program was written to estimate all possible reference variable rotations for three factors (i.e., nine variables taken three at a time yields 84 combinations; see Equation 24). A monotonic plot of the values $ln|\Lambda_s|$ values showed a nearly linear increase up to a final plateau with four good solutions. The reference variable set with the largest $ln|\Lambda_s|$ value was found for variables 2, 5 and 8. The structural equation model was refitted forcing the zero loadings on these three reference variables, and these results are listed as Model 4 of in Table 2. This model obtains exactly the same goodness of fit as Model 3, but here all free parameters are better behaved and the standard errors are uniformly smaller. These results suggest we might wish to add a few more loadings to our original simple structure solution — Variables 6, 7, and 9 also seem to load on factor 1, factor 2 remains the same, and variable 7 has a far smaller loading on factor 3.

These final reference variable results of Model 4 require only the minimal zero restrictions on the overall oblique solution and the non-zero parameters are well-behaved. The specific location of the zeros for these reference

Table 2

Numerical Results of REFERENCE VARIABLE Multiple Group Structural Equation Models Fitted to Meredith's Four Group Covariance Matrices

	Model 3 Initial References			Model 4 Rotated References		
[1]: **Λ** = Factor Loadings						
Variables	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$
1. Visual Perception	.77*	.0 =	.0 =	.73*	.15	.09
2. Rotating Cubes	.68*	-.14	-.18	.47*	.0 =	.0 =
3. Paper Form Board	.85*	-.11	-.24	.58*	.06	-.01
4. General Information	.0 =	.81*	.0 =	.13	.79*	-.06
5. Sentence Completion	-.38	1.01*	.24	.0 =	.90*	.0 =
6. Word Classification	-.09	.74*	.29	.28*	.69*	.08
7. Figure Recognition	.0 =	.0 =	.59*	.50*	-.03	.28*
8. Object-Number Pairs	-2.21*	.56	2.36*	.0 =	.0 =	.85*
9. Number-Figure Pairs	-.68	.18	1.08*	.30*	-.01	.43*
[2]: $\mathbf{\Delta}^{(g)}$ = Factor Standard Deviations						
Group	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$
1. Pasteur/Low	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =
2. Pasteur/High	.66*	.98*	.60*	.63*	.97*	1.10*
3. Grant/Low	.81*	.90*	.75*	.74*	.89*	.74*
4. Grant/High	.92*	.88*	.96*	.87*	.86*	1.04*
[3]: $\mathbf{R}^{(g)}$ = Factor Correlations						
Group	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
1. Pasteur/Low	.43*	.91*	.17	.11	.07	.08
2. Pasteur/High	.71*	.72*	.34	.38	-.32	.11
3. Grant/Low	.71*	.94*	.58*	.43*	.17	.49*
4. Grant/High	.59*	.91*	.37*	.33*	.22*	.19*
[4]: Goodness of Fit						
Parameters Estimated	$Npar_{model3} = 51$			$Npar_{model4} = 51$		
Degrees of Freedom	$df_{model3} = 129$			$df_{model4} = 129$		
Likelihood Ratio	$\chi^2_{model3} = 127.$			$\chi^2_{model4} = 127.$		

Note. All values are Maximum Likelihood Estimates from LISREL-7 (see Appendix). Models 3 and 4 estimated $\Psi_{m,m}$ = [.45*, .54*, .71*, .35*, .19*, .45*, .50*, .31, .68*].
* Free Parameter where $MLE_p > 2SE_p$. Labels "v =" is fixed at value v.

variables was determined empirically. This is desirable because we did not know the best location and we defined this position objectively. Strictly speaking, then, this means the resulting standard errors for these parameters are underestimated and should not be used to examine further test statistics (see Archer & Jennrich, 1973; Browne, personal communication, September, 1988; Jennrich, 1974; Lambert, Wildt, & Durand, 1991). These results demonstrate that the choice of reference variables alters the factor loadings and covariances but, since this model is exactly identified, this rotation does not change the common factor space or goodness-of-fit.

Result 2a: Orthogonal Group Rotations

Another way to start this Confactor estimation is by using a model where we estimate as many free factor loadings as possible. Model 5 of Table 3 shows the relaxation of the previous models to have a fully free (M x K) pattern $\mathbf{\Lambda}$. This full loading pattern model is identified by restricting only $\mathbf{\Phi}^{(1)} = \mathbf{I}$ and $\mathbf{\Phi}^{(2)} = \mathbf{\Delta}^2$. The numerical results of this model fit (using LISREL-7) verify several assertions of the previous section: (a) The goodness-of-fit of this model is the same as before (with χ^2=127 on DF=129), (b) the estimates of uniqueness $\mathbf{\Psi}$ are identical, and (c) all model parameters have well defined standard errors. Unfortunately, the empirical standard errors are relatively large and this overall solution remains slightly unstable. This model is also exactly-identified and is simply another a rotation of the previous oblique models of Table 2.

In Model 6 of Table 3 we try another practical variation on this basic theme. Here we require only the first group to have orthogonal factors with $\mathbf{\Phi}^{(1)} = \mathbf{I}$ but we allow all other $\mathbf{\Phi}^{(g)}$ to be freely estimated. To identify this model we need to place the minimal identification conditions on loadings $\mathbf{\Lambda}$. The standard orthogonality conditions for identification require a corresponding placement of at least $K(K - 1)/2$ fixed values within the loading matrix of Model 5. In Model 6 we fixed zero loadings for variable 1 factors 2 and 3 and for variable 4 on factor 3. These results show this model is exactly-identified, and again the goodness-of-fit is χ^2=127 on DF=129. In this case the standard errors are smaller and the model parameters are more stable.

The two models of Table 3 illustrate an alternative set of choices of reference groups and reference variables. Of course, slightly different results will emerge from different choices of the reference loadings and the specific reference group (where $\mathbf{\Phi}^{(g)}$ is diagonal or fixed). If these restrictions can be chosen on some a priori basis, then no further rotation is needed. The identical

Table 3

Numerical Results of ORTHOGONAL GROUP Multiple Group Structural Equation Models Fitted to Meredith's Four Group Covariance Matrices

	Model 5 Orthogonal 1 & 2			Model 6 Orthogonal 1 Only		
[1]: Λ = Factor Loadings						
Variables	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$
1. Visual Perception	.67*	.37*	-.12	.77*	.0 =	.0 =
2. Rotating Cubes	.43*	.14	-.13	.46*	-.09	-.06
3. Paper Form Board	.52*	.23	-.17	.58*	.05	-.09
4. General Information	-.06	.80*	-.10	.35*	.73*	.0 =
5. Sentence Completion	-.18	.88*	-.01	.27	.85*	.08
6. Word Classification	.14	.77*	-.01	.49*	.60*	.10
7. Figure Recognition	.55*	.16*	.13	.53*	-.14	.20*
8. Object-Number Pairs	.26	.13	.80*	.16	-.06	.83*
9. Number-Figure Pairs	.41*	.14	.33*	.37*	-.10	.38*
[2]: $\Delta^{(g)}$ = Factor Standard Deviations						
Group	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$
1. Pasteur/Low	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =
2. Pasteur/High	.51*	.98*	1.14*	.66*	.89*	1.13*
3. Grant/Low	.65*	.94*	.70*	.81*	.77*	.73*
4. Grant/High	.85*	.91*	.98*	.92*	.80*	1.01*
[3]: $R^{(g)}$ = Factor Correlations						
Group	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
1. Pasteur/Low	.0 =	.0 =	.0 =	.0 =	.0 =	.0 =
2. Pasteur/High	.0 =	.0 =	.0 =	.51*	-.15	.01
3. Grant/Low	.30	-.06	.27	.42*	.17	.38
4. Grant/High	.25	.21	.10	.18	.20	-.02
[4]: Goodness of Fit						
Parameters Estimated	$Npar_{model5} = 51$			$Npar_{model6} = 51$		
Degrees of Freedom	$df_{model5} = 129$			$df_{model6} = 129$		
Likelihood Ratio	$\chi^2_{model5} = 127.$			$\chi^2_{model6} = 127$		

Note. All values are Maximum Likelihood Estimates from LISREL-7 (see Appendix). Models 5 and 6 estimated $\Psi_{m,m}$ = [.45*, .54*, .71*, .35*, .19*, .45*, .50*, .31, .68*].
* Free Parameter where $MLE_p > 2SE_p$. Labels "v =" is fixed at value v.

goodness-of-fit is obtained under any choice of such restrictions. However, since these references variables and reference groups are usually arbitrary, this results lead to further models with alternative rotations.

Result 2b: External Orthogonal Rotations

As demonstrated, Models 2, 3, 4, and 5 all lead to the same goodness-of-fit and uniquenesses. These different loadings obtained are due to different rotations of the invariant common factors. As shown earlier, the basic orthogonal structure of the obtained loading matrices $\mathbf{\Lambda}$ is not altered by these choices of rotation, and this led to another set of rotational possibilities. First we obtained the invariant factor loading matrix from the results of Model 5. Second, we calculated the left orthonormal vectors $\mathbf{U}_\lambda$ from the singular value decomposition of this loading matrix (i.e., $\mathbf{\Lambda}=\mathbf{U}_\lambda\mathbf{S}_\lambda\mathbf{V'}_\lambda$). Third, this matrix $\mathbf{U}_\lambda$ was then rotated externally (e.g., using SAS routines) to maximize the orthogonal normal Varimax criteria (see Browne & DuToit, 1987). Fourth, $\mathbf{U}_\lambda$ was also rotated externally based on an oblique Procrustes criterion with a target defined by the very simple structure of Model 2. Other rotational procedures could be used here as well. The results of Table 4 give a structural equation extension of these models.

In Model 7 the loadings for three rows (reference variables 2, 5 and 8) were all fixed to be equal to their corresponding Varimax loadings. The required K^2 constraints were all placed in the $\mathbf{\Lambda}$ only to retain the scaling of the singular value $\mathbf{U}_\lambda$. These fixed K^2 constraints allow the estimation of parameters and standard errors for the rest of the invariant factor loadings in $\mathbf{\Lambda}$, the factor deviations $\mathbf{\Delta}^{(g)}$ and correlations $\mathbf{R}^{(g)}$ in all groups. The results of Model 7 give the maximum likelihood estimates and standard errors for all other parameters of the multiple group model, and the identical goodness-of-fit (χ^2=127 on DF=129) was obtained. These results include both (a) parallel proportional profiles and (b) a simple structure defined by Varimax.

In Model 8 these same fixed values were next forced to be equal to the values obtained from the Oblique Procrustes rotation. The orthonormal constraints on $\mathbf{U}_\lambda$ yield a limited form of obliquity, so the results were virtually the same and the identical goodness-of-fit (χ^2=127 on DF=129) was obtained. However, the results of both of these models suggests a remarkably simple patterning: The very simple structure of Model 2 is appropriate, except for variable 7 (Figure Recognition) which seems to load on both factors 1 and 3.

Table 4
Numerical Results of EXTERNAL ROTATIONS & MULTIPLE GROUP Structural Equation Model Fitted to Meredith's Four Group Covariance Matrices

	Model 7 Varimax Rotation			Model 8 Procrustes Rotation		
[1]: $\mathbf{\Lambda}$ = Factor Loadings						
Variables	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$
1. Visual Perception	.59*	.14*	.06	.61*	.07	.18
2. Rotating Cubes	.41 =	.02 =	-.02 =	.42 =	-.02 =	.06 =
3. Paper Form Board	.50*	.07	-.03	.51*	.02	-.06
4. General Information	.01	.58*	-.07	.04	.57*	-.08
5. Sentence Completion	-.13 =	.67 =	-.01 =	-.10 =	.66 =	-.04 =
6. Word Classification	.12	.52*	.06	.15	.49*	.08
7. Figure Recognition	.37*	.01	.27*	.37*	-.04	.34*
8. Object-Number Pairs	-.22 =	.03 =	.86 =	-.21 =	.02 =	.82 =
9. Number-Figure Pairs	.15	.02	.43*	.16	-.01	.46*
[2]: $\mathbf{\Delta}^{(g)}$ = Factor Standard Deviations						
Group	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$
1. Pasteur/Low	1.40*	1.05*	1.14*	1.11*	1.42*	1.06*
2. Pasteur/High	1.37*	1.05*	.68*	.78*	1.39*	1.05*
3. Grant/Low	1.29*	.77*	.82*	.83*	1.31*	.79*
4. Grant/High	1.25*	1.11*	.98*	.95*	1.27*	1.11*
[3]: $\mathbf{R}^{(g)}$ = Factor Correlations						
Group	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
1. Pasteur/Low	.12	.21	.36*	.29	.20	.17
2. Pasteur/High	.12	.39	-.15	.46*	-.30	.18
3. Grant/Low	.55*	.48*	.51	.52*	.32	.59*
4. Grant/High	.27	.42*	.44*	.48*	.27	.31*
[4]: Goodness of Fit						
Parameters Estimated	$Npar_{model7} = 51$			$Npar_{model8} = 51$		
Degrees of Freedom	$df_{model7} = 129$			$df_{model8} = 129$		
Likelihood Ratio	$\chi^2_{model7} = 127.$			$\chi^2_{model8} = 127.$		

Note. All values are Maximum Likelihood Estimates from LISREL-7 (see Appendix). Models 7 and 8 estimated $\mathbf{\Psi}_{m,m}$ = [.45*, .54*, .71*, .35*, .19*, .45*, .50*, .31, .68*]; Singular Value Decomposition of $\mathbf{\Lambda} = \mathbf{U}_\lambda \mathbf{S}_\lambda \mathbf{V}_\lambda'$ produced $U_{m,1}'$ = [.38, .19, .27, .43, .45, .47, .24, .17, .20], $U_{m,2}'$ = [-.31, -.22, -.24, .37, .46, .19, -.38, -.39, -.34], $U_{m,3}'$ = [.33, .28, .35, -.03, -.18, -.04, .05, -.77, -.21].
* Free Parameter where $MLE_p > 2SE_p$.

There are several other ways to use the results of these kinds of external rotations. For example, we could also fit a structural model where, say, the values of the K - 1 largest negative loadings of each column of the oblique solution have been forced to be zero. This is another rotation which would be consistent with our hypothesized pattern, and this model obtains the same goodness-of-fit and uniquenesses are before. On the other hand, if we are satisfied with the invariant result we can calculate standard errors directly from the external rotation using the techniques of Archer and Jennrich (1973). Finally, if we want to optimize a specific rotational criterion (e.g., normal Varimax), we can require this directly from the initial structural equation estimation and obtain the standard errors directly (as in Browne & DuToit, 1987).

Result 3a: Adding Latent Path Constraints

The previous discussion suggests several ways to add additional restrictions on the factor covariances $\mathbf{\Phi}^{(g)}$ and, correspondingly, require less restrictions required on the factor loadings $\mathbf{\Lambda}$. Following Cattell's (1966) suggestions we explore another set of structural model restrictions listed in Table 5.

In Model 9 of Table 5 we estimate a $K = 3$ common factor solution as before but this loading matrix $\mathbf{\Lambda}$ contains only K zero restrictions (i.e., three fixed zeros on variables 4 and 8). Within this model we also fit one second order factor with loadings $\mathbf{B}$ and these loadings are required to be invariant over groups. This model is identified here by: (a) requiring invariance of the second order loadings over all groups, (b) fixing the scale of the second order factor at 1 in the first group, (c) fixing the scale of the three second order unique factors at specific values. These three unique variances of Model 9 were fixed to estimate the $\mathbf{\Lambda}$ where the first order factors have unit variance in the first group. (This scaling was accomplished by initially setting these values at 1, estimating the solution, standardizing the results, and rerunning the same program with the necessary standardized deviations; i.e., .96=, .83=, .94=). The second order loadings $\mathbf{B}$ = [.29, .55, .34] obtained are invariant over groups but the second order common and unique variances change over groups. This model fits these data well with χ^2=131 with DF=132, but these new model parameters are relatively unstable.

Model 10 of Table 6 adds a few restrictions to the previous model. In addition to restrictions (a), (b) and (c) listed above, we also (d) fix the second order uniquenesses on this first factor to be zero (rather than .93), (e) restrict the second order loading on the first factor to be equal to unity, and (f) restrict

Table 5

Numerical Results of SECOND ORDER PATHS Multiple Group Structural Equation Models Fitted to Meredith's Four Group Covariance Matrices

	Model 9 Invariant Loadings			Model 10 Path Restrictions		
[1]: **Λ** = First Order Factor Loadings						
Variables	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$
1. Visual Perception	.69*	.29	-.06	.71*	.38*	-.05
2. Rotating Cubes	.48*	.09	-.08	.44*	.15	-.08
3. Paper Form Board	.54*	.17	-.11	.52*	.24*	-.08
4. General Information	.0 =	.82*	-.14	.0 =	.81*	.06
5. Sentence Completion	-.17	.93*	-.06	-.16	.90*	.16*
6. Word Classification	.15	.76	-.03	.18*	.77*	.11
7. Figure Recognition	.51*	.06	.19	.53*	.14	.19
8. Object-Number Pairs	.0 =	.0 =	.88*	.0 =	.0 =	1.07*
9. Number-Figure Pairs	.31*	.05	.35*	.35*	.11	.26*
[2]: **B** = Second Order Path Loadings						
Variables	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$	$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$
1. Second Order η	.29	.55	34	1.0=	.08	.33
[3]: $\Psi^{(g2)}$ = First Order Unique Deviations						
Group	$\psi_{1,1}$	$\psi_{2,2}$	$\psi_{2,3}$	$\psi_{1,1}$	$\psi_{2,2}$	$\psi_{2,3}$
1. Pasteur/Low	.96 =	.83 =	.94 =	.0 =	.99 =	.95 =
2. Pasteur/High	.47*	.84*	1.07*	.0 =	.97*	1.08*
3. Grant/Low	.59*	.54	.60*	.0 =	.87*	.81*
4. Grant/High	.77*	.56	.95*	.0 =	.86*	.97*
[4]: $\Delta^{(g2)}$ = Second Order Standard Deviations						
Group	δ_1			δ_1		
1. Pasteur/Low	1.0 =			1.0 =		
2. Pasteur/High	.87			.52*		
3. Grant/Low	1.36			.67*		
4. Grant/High	1.30			.89*		
[5]: Goodness of Fit						
Parameters Estimated	$Npar_{model9} = 54$			$Npar_{model10} = 58$		
Degrees of Freedom	$df_{model9} = 132$			$df_{model10} = 136$		
Likelihood Ratio	$\chi^2_{model9} = 132.$			$\chi^2_{model10} = 135$		

Note. All values are Maximum Likelihood Estimates from LISREL-7 (see Appendix). Models 9 estimated $\Psi_{m,m}$ = [.45*, .54*, .72*, .36*, .18*, .45*, .49*, .24, .69*], and Models 10 estimated $\Psi_{m,m}$ = [.45*, .55*, .73*, .36*, .18*, .45*, .50*, -.12, .73*]).

* Free Parameter where $MLE_p > 2SE_p$.

the second order uniquenesses on the first factor to be equal to zero. These restrictions have the effect of making the factor 1 equivalent to the second order factor so we can reinterpret the invariant loadings **B** as invariant latent regressions from factor 1 to factors 2 and 3. This model requires a invariant but patterned covariance matrix for each group. The result is a good fit of χ^2=135 on DF=136 with more stable first order loadings, but the second order coefficients are small and one unstable uniqueness results (i.e., $\psi_{8,8}$ = -.12).

There are many over-identified models that may be examined from any of the previous exactly-identified rotations. One obvious alternative allows a free but invariant loadings Λ with all group covariance matrices orthogonal but proportional across all groups (i.e., $\Phi^{(1)} = \mathbf{I}$, $\Phi^{(g)} = \Delta^{(g)} \Delta^{(g)}$, g = 2 to G). The model obtains an χ^2 = 131 on DF = 135 so the difference due to overall orthogonality is $d\chi^2$ =4 on dDF = 6 and this hypothesis is reasonable. Another over-identified model which can be fitted is one where factor loadings and correlations are invariant over groups (i.e., $\mathbf{R}^{(g)}$=$\mathbf{R}$; as in oblique PARAFAC analysis by Harshman & Lundy, 1984). These restrictions obtained an χ^2=129 on DF=132, but this model proved to be relatively difficult to estimate with these data although, in theory, this model is over-identified (see McDonald, 1984).

Result 3b: Adding Latent Means

One clear advantage of the metric invariance model is the further examination of latent variable means. Table 6 gives results for two final models which include restrictions on the multiple group latent mean vectors $\Theta^{(g)}$ and are fitted to all group covariance matrices and mean vectors simultaneously (see Appendix 2). Both models are identified here by: (1) adding the constraint of invariant unique means (or variable intercepts υ) in a new fourth column of Λ, and (2) forcing the latent mean vector $\Theta^{(1)} = \mathbf{O}$ in group 1. These constraints are consistent with the way we have rescaled the mean vectors (means scaled to be zero in group 1; see Appendix 1). The key difference between Model 11 and Model 12 is in the patterning of the free and fixed loadings of the invariant Λ.

In Model 11 of Table 6 we estimate latent variable means using the reference variable Λ or Model 4 (in Table 2). This model obtains stable estimates with the invariant Λ, and several group differences are seen in the latent means $\Theta^{(g)}$: There are no mean differences on factor 1, large mean differences on factor 2, and notable mean differences on factor 3. Again, we notice one unstable uniqueness (i.e., $\psi_{8,8}$ = -.10). The goodness-of-fit of this three factor hypothesis for the multiple group covariances and means is χ^2=173 on DF=147. By contrast to the previous covariance Model 4 (χ^2=127 on DF=129), the loss of fit due to including latent means is $d\chi^2$=46 on dDF=18.

In Model 12 of Table 6 we estimate latent variable means $\Theta^{(g)}$ using the very simple structure Λ of Model 2 (in Table 1). The means vectors of this model are identified in the same way as the previous Model 11. Once again, this model obtains stable estimates with the invariant Λ, and several group differences are seen in the latent means $\Theta^{(g)}$: There are no mean differences on factor 1, large mean differences on factor 2, and now no mean differences on factor 3. The goodness-of-fit of this three factor hypothesis for the multiple group covariances and means is χ^2=226 on DF=159. By contrast to the previous covariance Model 2 (χ^2=173 on DF=141), the loss of fit due to including latent means is $d\chi^2$=53 on dDF=18.

The direct comparison between these final two models can be important in practice. The differences in overall fit test between Models 11 and 12 are $d\chi^2$=53 on dDF=12, and this is an index of the improvement in fit due to the 12 additional loadings of Model 11. The previous covariance-based comparison of the same loading patterns (i.e., Model 4 versus Model 2) yielded a difference of $d\chi^2$=46 on dDF=12. The additional $d\chi^2$=7 indexes the misfit in the mean differences due to the requirement of the very simple structure Λ. Perhaps more important is the fact that under Model 11 assumptions we find notable mean differences on factor 3 whereas under Model 12 assumptions we find similar mean differences are not large. Factor 3 is a slightly different factor in these two models, and this may in practice lead to different substantive conclusions about group differences.

Discussion

Structural Confactor Solutions

This research has emphasized the use of Cattell's (1944) rotational principles in the context of contemporary structural equation modeling. In accordance with Cattell's earlier logic, we have shown how invariance restrictions over groups helps make the factor model unique and estimable. The surprising result is an entire set of loadings Λ may be estimated with restrictive conditions on the covariances $\Phi^{(g)}$ for only two preselected groups. These same results also show how additional rotational principles beyond invariance are needed to make a factor model unique to any arbitrary selection of groups. In this sense, Confactor analysis can be defined as a two-

Table 6

Numerical Results of LATENT MEANS Multiple Group Structural Equation Models Fitted to Meredith's Four Group Covariance Matrices

	Model 11 Reference & Means			Model 12 Very Simple & Means		
[1]: $\mathbf{\Lambda}$ = Factor Loadings						
Variables	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$	$\lambda_{m,1}$	$\lambda_{m,2}$	$\lambda_{m,3}$
1. Visual Perception	.73*	.08	.09	.79*	.0 =	.0 =
2. Rotating Cubes	.46*	.0 =	.0 =	.41*	.0 =	.0 =
3. Paper Form Board	.56*	.04	.04	.59*	.0 =	.0 =
4. General Information	.0 =	.84*	.0 =	.0 =	.83*	.0 =
5. Sentence Completion	-.05	.81*	.05	.0 =	.78*	.0 =
6. Word Classification	.18*	.77*	.08	.0 =	.83*	.0 =
7. Figure Recognition	.49*	.03	.21*	.0 =	.0 =	.44*
8. Object-Number Pairs	.0 =	.0 =	1.04*	.0 =	.0 =	.59*
9. Number-Figure Pairs	.32*	.04	.34*	.0 =	.0 =	.62*
[2]: $\mathbf{\Theta}^{(g)}$ = Factor Means						
Group	θ_1	θ_2	θ_3	θ_1	θ_2	θ_3
1. Pasteur/Low	.0 =	.0 =	.0 =	.0 =	.0 =	.0 =
2. Pasteur/High	-.02	.38*	.39*	.04	.39*	.34
3. Grant/Low	.17	.84*	-.43*	.09	.83*	-.26
4. Grant/High	.17	1.12*	.03	.07	1.20*	.41
[3]: $\mathbf{\Delta}^{(g)}$ = Factor Standard Deviations						
Group	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{1,1}$	$\delta_{2,2}$	$\delta_{2,3}$
1. Pasteur/Low	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =	1.0 =
2. Pasteur/High	.62*	.96*	1.10*	.72*	.97*	.88*
3. Grant/Low	.74*	.89*	.85*	.80*	.91*	.70*
4. Grant/High	.88*	.89*	1.04*	.86*	.91*	1.06*
[4]: $\mathbf{R}^{(g)}$ = Factor Correlations						
Group	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
1. Pasteur/Low	.22	.07	-.03	.36*	.47*	.22*
2. Pasteur/High	.48*	-.29	.02	.60*	.05	.21
3. Grant/Low	.54*	.05	.32*	.70*	.79*	.72*
4. Grant/High	.42*	.12	.11	.56*	.86*	.31
[5]: Goodness of Fit						
Parameters Estimated	$Npar_{model11} = 79$			$Npar_{model12} = 57$		
Degrees of Freedom	$df_{model11} = 147$			$df_{model12} = 159$		
Likelihood Ratio	$\chi^2_{model11} = 173.$			$\chi^2_{model12} = 226.$		

Note. All values are Maximum Likelihood Estimates from LISREL-7 (see Appendix). Model 11 estimated $\mathbf{\Psi}_{m,m}$ = [.46*, .55*, .71*, .32*, .27*, .43*, .52*, -.10, .74*], and $\mathbf{\Lambda}_{4,m}$' = [-.06, -.06, -.01, .06, -.09, .07, .13, .00, -.02]. Model 12 estimated $\mathbf{\Psi}_{m,m}$ = [.43*, .57*, .69*, .32*, .29*, .44*, .57*, .76*, .58*], and $\mathbf{\Lambda}_{4,m}$' = [.01, -.05, .02, .06, -.09, .04, .13, -.07, -.05].

* Free Parameter where $MLE_p > 2SE_p$.

step criterion: *Factor loading invariance is the primary criterion for unique estimation and a secondary criterion is needed for unique rotation.*

These overall results conform with most of the previous research on the Confactor problem. We have formalized and made explicit the heretofore implicit constraints of the orthogonal Confactor solution. These results suggested the number and nature of additional constraints for the partially oblique case, the freely oblique case, and the higher order invariance model. These results show that oblique Confactor rotation is not in and of itself unique, but it is a step in the specific direction, and it can be parsimonious. The Confactor model is a reduction of the uncertainty of a large infinity of solutions for many groups to a smaller infinity of solutions for one group. This reduction in uncertainty increases with an increase in the number of groups.

We have suggested some practical ways to implement a fully oblique Confactor solution using the widely available structural equation programs, such as LISREL. One approach starts with an exactly-identified multiple group solution, calculates all possible reference variable solutions, and refits the model using empirically selected reference variables. A second approach starts with a multiple group exactly-identified solution, uses external rotation criteria to determine a minimal hyperplane, and refits the model using empirically selected reference variables. In a third approach we have used higher-order or latent path invariance restrictions and a minimal factor patterning, and we have examined the possibility of using latent factor means as well. All of these oblique Confactor solutions can be estimated within available structural programs and all can be tested against alternative simple structure alternatives.

Future solutions can count on the use of advances in structural equation algorithms. For example, the simultaneous approach described earlier is technically superior to the two-stage approach but it now requires special programming which is tedious at best. McDonald (1980) discussed the use of additional model constraints as "non-standard" or "scalar" constraints. McDonald also detailed the basic features of numerical programming required for their addition and the COSAN program (Fraser & McDonald, 1988) allows for the inclusion of extra constraints on a factor pattern. These main problem is these constraints require the calculation of the first order partial derivatives of the parameters with respect to the fitting function. Browne and du Toit (1987) have recently described a new programming approach (AUFIT) which allows these numerical options to be more easily accessible. Other possibilities include using EM-Based algorithms (see Tisak & Meredith, 1989), relaxed identification algorithms (Shapiro, 1986), and Bootstrap techniques (e.g., Lambert et al, 1991).

Substantive Issues

We have also pointed out how substantive issues may dictate a specific solution. For example, within these data the simple structure analysis (of Table 1; Jöreskog, 1971) has been widely accepted. In the simple structure context these factors have been labelled Spatial, Verbal, and Memory, and widely used. Jöreskog (1971) concluded:

> Altogether these results suggest two alternative descriptions of the data. One is that the whole factor structure is invariant over populations with a three-factor solution of a fairly complex form. The other is to represent the tests in each population by three factors of a particularly simple form, but these factors have different variance-covariance matrices in different populations. Additional studies with larger sample sizes are needed to discriminate statistically between the two models. Perhaps the second alternative has the most intuitive appeal... (Jöreskog, 1971, p.205).

Our reanalyses here suggests this very simple structure solution is not necessarily indicated by either these parameter estimates, this goodness-of-fit, or the original theory. At very least one complex variable (e.g., Figure Recognition) is apparent in these data, and perhaps other variables are complex as well (e.g., 6 and 9). This lack of simple structure makes it more difficult to label the factors but this problem is compensated by the invariance of the factor pattern over groups. These kinds of alternatives to simple structure were recognized by both Meredith (1964b) and Jöreskog (1971), and many others (e.g., Butler, 1969; Nesselroade, 1983; Overall, 1964).

We still allow for a sequential determination of the invariant factor pattern using statistical bases. Other models can be fitted and compared by a strict adherence to the likelihood ratio statistics. On the other hand, factor invariance and Confactor resolution is an empirical property of data and, as such, it is not a necessary result. We always need to ask "how many factors do we need if we want an invariant model" versus "how many parameters do we need to define a non-invariant model" (see Horn et al, 1983; Horn & McArdle, 1992). At some point statistical comparisons will become moot and we will be forced to choose between models on other grounds. As we have demonstrated here, two models can fit the same data equally well, and be indistinguishable in terms of the data, but the parameters of one model may be more interpretable than another. This brings us back to the traditional problems of choosing rotations in exploratory factor analysis. Here, as always, a good selection of K very simple structure variables are needed to serve as reference variables in any solution.

Our use of higher order or latent path models are probably not indicated with these data, but these models illustrate several reasonable structural possibilities. First, an oblique model was estimated with the minimum restrictions on $\mathbf{\Lambda}$ usually required for the orthogonal model. This is one illustration of a tradeoff between restrictions on $\mathbf{\Lambda}$ and on $\mathbf{\Phi}$. These partial invariance restrictions on $\mathbf{\Phi}^{(g)}$ are not enough to ensure identification (i.e., the K restrictions on $\mathbf{\Lambda}$ are still required for factor separation). Nevertheless, both latent path models (9 and 10) do (a) require an invariant first order factor pattern $\mathbf{\Lambda}$, and (b) permit a completely different set of correlations $\mathbf{R}^{(g)}$ among the first order factors. Second, these expressions of model constraints follow a reasonable generalization of the selection theorems of Meredith (1964a). That is, these constraints can be seen to follow Meredith's selection model reapplied at the second order or latent path level. These models allow a variety of useful ways to estimate an invariant factor pattern with minimal simple structure constraints.

One caution that needs to be expressed is the avoidance of hypotheses without the needed substantive basis. The PARAFAC model fixes the rotation of $\mathbf{\Lambda}$ by making several overidentifying restrictions on the covariances but this creates two concerns. First, the equality restrictions on all the correlation matrices (**R**) quickly lead to an over-identified restrictions in many groups. Unlike the oblique Confactor model, the number of restrictions required in this model increases with the number of groups used. Second, the original basis of Cattell's (1944; Cattell & Cattell, 1955) models as well as the selection theory used by Meredith (1964a) argue directly against the use of equal correlations among factors. That is, unless the groups are *random* samples from the same population, there is little statistical basis for the equal correlation hypothesis. McDonald (1984) has examined the identification status of this oblique form of the PARAFAC model and he criticized its usage on similar counts. Models with restrictions on the higher order factors, on the latent paths, on the latent variances, or on the latent means, can be used to identify the factor loadings $\mathbf{\Lambda}$, but these restrictions will be of most benefit if they have substantive relevance.

Cattell (1966, 1972) suggested the use of constraints on both means and covariances, and Meredith (1990) recently demonstrated the need for using cross-products. (Model 11 is an example of this approach). Another appealing situation where these cross-product models can be useful comes in the analysis of repeated measures problems (see McArdle, 1984b, 1988; McDonald, 1985; Nesselroade, 1983; Rozeboom, 1977). In fully designed multivariate experiments we may be able to introduce the necessary mathematical restrictions required for an unambiguous oblique Confactor rotation. In the examples presented here we could have placed more restrictions on the models to represent hypotheses about the two schools and the selection mechanism used by Meredith (1964b). At very least, we can examine these kinds of covariance and mean changes from an experimental point of view.

Factorial Invariance and Structural Modeling

Many contemporary structural equation models end up with a very simple structure pattern because no alternatives are used. What needs to be recognized is invariance of the factor pattern is often more critical than complexity of the factor scores. Often these differences cannot be understood by differences in goodness-of-fit and other comparisons need to be made. At another level our higher order models lead to a variety of other options which can now be examined. These invariance principles are even more useful with more groups, more measures, more strata, and more experimental design features. These Confactor principles also generalize to all forms of structural equation models, including latent path and analysis of covariance structures. Parsimony in terms of the total number of parameters required is a key to effective structural equation modeling, and invariance over groups is one of the most powerful structural modeling devices.

To be sure, structural Confactor analysis does not guarantee the resolution of an invariant factor model; The invariant but otherwise unrestricted K-factor model might not fit the multiple group observations. Instead, the key benefit of the Confactor model is the possibility of directing research towards finding invariant parameters within a structural system. This is not a new goal for structural modeling:

> The identification of relationships that remain invariant among variables under different conditions and transformations is a major goal of empirical research ... Demonstration of factor invariance is one particular realization of a major goal of science — namely the identification of invariant relationships. The invariant relationships involved are those between factors (unobserved or latent variables) and observed variables or, in higher order analyses, other factors. At the first order of analysis for example, factor invariance signals a kind of constancy of a measurement system and thus the reasonableness of comparing phenomena in quantitative rather than in qualitative terms ... (from J.R. Nesselroade, 1983, pp. 59, 62-63).

In summary, we have shown how the old principles of Confactor can be successfully applied using the new structural equation techniques like

LISREL. Not all datasets, even those carefully selected, should be thought to have a very simple structure basis. In structural Confactor analysis we search for an invariant factor pattern and we use the new algorithms to accomplish this goal. The Confactor model allows simple structure to emerge, but it does not demand it. The original ideas of Cattell (1944) and Meredith (1964a) are now practically useful devices for functionally useful structural equation models.

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Appendix 1
MEREDITH.MOM: Multiple Group Labels, Correlations Rescaled Deviations, and Rescaled Means (Meredith, 1964b)

```
Group 1: N = 77 Pasteur School with Low Addition Scores
'pl-vis','pl-cub','pl-pap','pl-inf','pl-sen','pl-wor','pl-fig','pl-
obj','pl-num'
1.00
 .32    1.00
 .48     .33    1.00
 .28     .01     .06     1.00
 .26     .01     .01      .75    1.00
 .40     .26     .10      .60     .63    1.00
 .42     .32     .22      .15     .07     .36    1.00
 .12     .05     .03     -.08     .06     .19     .29    1.00
 .23    -.04     .01     -.05     .10     .24     .19     .38    1.00
1.0000  1.0000  1.0000   1.0000  1.0000  1.0000  1.0000  1.0000  1.0000
0       0       0        0       0       0       0       0       0

Group 2: N = 79 Pasteur School with High Addition Scores
'ph-vis','ph-cub','ph-pap','ph-inf','ph-sen','ph-wor','ph-fig','ph-
obj','ph-num'
1.00
 .24    1.00
 .23     .22    1.00
 .32     .05     .23     1.00
 .35     .23     .18      .68    1.00
 .36     .10     .11      .59     .66    1.00
 .22     .01    -.07      .09     .11     .12    1.00
-.02    -.01    -.13      .05     .08     .03     .19    1.00
 .09    -.14    -.06      .16     .02     .12     .15     .29    1.00
0.9054  0.7143  0.9655   0.9322  1.0000  1.0192  0.8636  1.1064  0.9565
0.0946 -0.1964  0.0690   0.4237  0.2308  0.3077  0.1818  0.4043  0.0435

Group 3: N = 74 Grant-White School with Low Addition Scores
'gl-vis','gl-cub','gl-pap','gl-inf','gl-sen','gl-wor','gl-fig','gl-
obj','gl-num'
1.00
 .34    1.00
 .41     .21    1.00
 .38     .32     .31     1.00
 .40     .16     .24      .69    1.00
 .42     .13     .35      .55     .65    1.00
 .35     .27     .30      .17     .20     .31    1.00
 .16     .01     .09      .31     .30     .34     .31    1.00
 .35     .27     .09      .34     .27     .27     .38     .38    1.00
0.8919  0.8571  0.8966   0.9576  0.9038  0.9615  0.6932  0.8298  0.8478
0.0676  0.0536  0.0345   0.7119  0.5962  0.7308  0.2614 -0.4468 -0.2174
```

```
Group 4: N = 71 Grant-White School with High Addition Scores
'gh-vis','gh-cub','gh-pap','gh-inf','gh-sen','gh-wor','gh-fig','gh-
obj','gh-num'
1.00
 .32    1.00
 .34     .18    1.00
 .31     .24     .31     1.00
 .22     .16     .29      .62    1.00
 .27     .20     .32      .57     .61     1.00
 .48     .31     .32      .18     .20      .29     1.00
 .20     .01     .15      .06     .19      .15      .36     1.00
 .42     .28     .40      .11     .07      .18      .35      .44    1.00
0.9730  0.7143  1.0345   0.9746  0.8654   1.0577   0.8409   1.0426 1.0217
0.0135  0.0536  0.1034   1.1186  0.7308   1.1346   0.3068   0.0426 0.3043
```

Appendix 2
Prototype LISREL-7 Program for Initial Confactor Estimation

```
Reference Group Solution: Group 1: N = 77 Pasteur School with Low
Addition

da ng=4 ni=9 no=77 ma=cm
la file=meredith.mom
km sy file=meredith.mom
sd file=meredith.mom
me file=meredith.mom
mo ny=9 ne=8 be=fi,fu ps=fi,sy ly=fi,fu te=fi,di
le
'factor1','factor2','factor3','higher','d-fac1','d-fac2','d-fac3','d-high'
pa ly
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
1 1 1 0 0 0 0 0
pa be
0 0 0 0 1 0 0 0
0 0 0 0 0 1 0 0
0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
pa ps
```

```
0
0 0
0 0 0
0 0 0 0
0 0 0 0 0
0 0 0 0 1 0
0 0 0 0 1 1 0
0 0 0 0 0 0 0 0
pa te
 1 1 1 1 1 1 1 1 1
st .5 all
fix be 1 5 be 2 6 be 3 7
st 1 be 1 5 be 2 6 be 3 7
st 1 ps 5 5 ps 6 6 ps 7 7
st 0 ps 5 6 ps 5 7 ps 6 7
fix ps 5 6 ps 5 7 ps 6 7
ou ns se tv pt pc fd mi
```

Group 2: N = 79 Pasteur School with High Addition Scores

```
da no=79
la file=meredith.mom
km sy file=meredith.mom
sd file=meredith.mom
me file=meredith.mom
mo be=ps ps=ps ly=in te=in
fi be 1 5 be 2 6 be 3 7
fr ps 5 6 ps 5 7 ps 6 7
ou
```

Group 3: N = 74 Grant-White School with Low Addition Scores

```
da no=74
la file=meredith.mom
km sy file=meredith.mom
sd file=meredith.mom
me file=meredith.mom
mo be=ps ps=ps ly=in te=in
fr be 1 5 be 2 6 be 3 7
fr ps 5 6 ps 5 7 ps 6 7
ou
```

Group 4: N = 71 Grant-White School with High Addition Scores

```
da no=71
la file=meredith.mom
km sy file=meredith.mom
sd file=meredith.mom
me file=meredith.mom
mo be=ps ps=ps ly=in te=in
fr be 1 5 be 2 6 be 3 7
fr ps 5 6 ps 5 7 ps 6 7
ou
```

Accepted 1989