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# Asymmetric Multidimensional Scaling of Two-Mode Three-Way Proximities

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**Abstract:** An asymmetric multidimensional scaling model and an associated nonmetric algorithm to analyze two-mode three-way proximities (object × object × source) are introduced. The model consists of a common object configuration and two kinds of weights, i.e., for both symmetry and asymmetry. In the common object configuration, each object is represented by a point and a circle (sphere, hypersphere) in a Euclidean space. The common object configuration represents pairwise proximity relationships between pairs of objects for the 'group' of all sources. Each source has its own symmetry weight and a set of asymmetry weights. Symmetry weights represent individual differences among sources of data in symmetric proximity relationships, and asymmetry weights represent individual differences among sources in asymmetric proximity relationships. The associated nonmetric algorithm, based on Kruskal's (1964b) nonmetric multidimensional scaling algorithm, is an extension of the algorithm for the asymmetric multidimensional scaling of one mode two-way proximities developed earlier (Okada and Imaizumi 1987). As an illustrative example, we analyze intergenerational occupational mobility from 1955 to 1985 in Japan among eight occupational categories.

**Keywords:** Asymmetric proximity; Multidimensional scaling; Two-mode three-way data.

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## 1. Introduction

Typical two-mode three-way proximities data consist of two or more symmetric proximity matrices, where each comes from an individual subject, experimental condition, or other source of data. Each element of a constituent proximity matrix represents a proximity between two objects corresponding to a given row and column of the proximity matrix. After the seminal work of Tucker and Messick (1963), several multidimensional scaling (MDS) models based on weighted Euclidean distance models were introduced to analyze two-mode three-way proximities (Bloxom 1968; Carroll and Chang 1970; Horan 1969; Schönemann 1972). These models attempt to account for differences in pairwise proximity relationships among sources or among proximity matrices, and are generally called individual differences MDS models. The differences have been customarily called "individual differences." Hereinafter that term refers to differences in pairwise proximity relationships depicted among sources. More general and extended models were also introduced (Carroll 1972; Lingoes and Borg 1978; Okada and Imaizumi 1980; Takane, Young, and de Leeuw 1977), but explicitly or implicitly such individual differences models assumed each of the square proximity matrices to be symmetric and thus cannot cope with a set of square *asymmetric* proximity matrices.

In the case of one-mode two-way proximities, most MDS models assume that a square proximity matrix can be regarded as symmetric (Guttman 1968; Kruskal 1964a, 1964b; Torgerson 1952). When a square proximity matrix is asymmetric, the easiest way to deal with the asymmetry of a proximity matrix to average the two conjugate proximities to obtain a square symmetric proximity matrix, and then to analyze the resulting symmetric proximity matrix by standard MDS (Harshman 1978). But simply ignoring or neglecting the asymmetry in a proximity matrix discards potentially valuable information. Several MDS models and associated algorithms have been introduced to deal with one-mode two-way asymmetric proximities (Chino 1978; Chino and Shiraiwa 1993; Constantine and Gower 1978; Cunningham 1978; DeSarbo and Manrai 1992; Gower 1977; Harshman 1978; Harshman, Green, Wind, and Lundy 1982; Krumhansl 1978; Okada and Imaizumi 1987; Saito 1991; Tobler 1977, 1979; Weeks and Bentler 1982; Young 1975, 1987; Zielman 1991; Zielman and Heiser 1993, 1996).

In the case of two-mode three-way proximities, which consist of two or more square asymmetric proximity matrices, asymmetry has been ignored when INDSCAL (INdividual Differences SCALing; Carroll and Chang 1970) or its derivative models were utilized. DeSarbo and Manrai (1992) and Zielman and Heiser (1993) noted the desirability of dealing with asymmetry in two-mode three-way proximities. Some models and associated algorithms were introduced to deal with asymmetry in two-mode three-way proximities

(DeSarbo, Johnson, A. Manrai, L. Manrai, and Edwards 1992; Zielman 1991; Zielman and Heiser 1991, 1993). For the case of two-mode three-way proximities, two sorts of individual differences should be noted: (a) individual differences in symmetric proximity relationships, and (b) individual differences in asymmetric proximity relationships.

The model of DeSarbo et al. (1992) is based on Tversky's (1977) "contrast model." The configuration of objects given by their approach represents a symmetric proximity relationship common to all sources, but ignores both asymmetric proximity relationships and individual differences in symmetric proximity relationships (which are gauged by other parameters). Individual differences in asymmetric proximity relationships are represented by still other parameters which have no association with the configuration of objects. We have to interpret separately the obtained configurations for objects and for parameters which represent asymmetry. Zielman and Heiser (1991, 1993) introduced the "slide vector" model, for one-mode two-way asymmetric proximities. In that model, the asymmetry is represented by a slide vector imbedded in a configuration of objects and is linked to the dimensions of the configuration rather than to its objects. The configuration of objects represents the symmetric proximity relationship. One has to add or subtract the slide vector in the configuration of objects to ascertain the asymmetric pairwise proximity relationships among objects. Although asymmetric proximity relationships are represented by projections of object vectors onto the slide vector, the projection corresponds to the square of the corresponding proximity value. It is not easy to visually scrutinize the asymmetry for each object pair.

Zielman and Heiser (1991, 1993) extended the model to deal with two-mode three-way proximities by introducing a set of dimension weights for each source, to represent the salience of each dimension for the source. The set of weights represents individual differences in both symmetric and asymmetric proximity relationships. The two different sorts of individual differences noted earlier are difficult to distinguish in the extended model. Zielman (1991) further extended his asymmetric MDS for one-mode two-way proximities to cope with two-mode three-way proximities by introducing an idiosyncratic configuration of objects for each source (or a set of transforming matrices applied to a common space) and by applying a set of weights to the parameters which represent asymmetry for each source. The set of idiosyncratic configurations of objects represents individual differences in symmetric proximity relationships, and the sets of weights represent individual differences in asymmetric proximity relationships.

The extended model can represent a symmetric proximity relationship for each source in a configuration, and an asymmetric proximity relationship can be represented both in the configuration of objects for each source or in another configuration. Although a symmetric element of proximity for any given pair of objects is represented as a distance in a configuration, the proximity itself is not represented as a distance in any configuration. From these considerations, three characteristics seem important in the development of asymmetric MDS model for two-mode three-way proximities: (a) symmetric and asymmetric proximity relationships are simultaneously represented in a configuration of objects, (b) individual differences in both symmetric and asymmetric proximity relationships are distinguished, and (c) the model is based on distance, where both symmetric and asymmetric aspects of any pairwise proximity as well as the proximity value itself are represented as distances in the configuration of objects.

Characteristic (a) leads to ease of interpretation of the result. When symmetric and asymmetric relationships are not represented in the same configuration, we must juxtapose symmetric and asymmetric relationships that are represented separately. It is far preferable to have both types of relationships in the same configuration. Characteristic (b) seems very significant because the two sorts of individual differences stem from different substantive causes. Characteristic (c) is important because most researchers utilizing MDS are accustomed to a model where an object is represented as a point and the recovered distance between two objects by an interpoint distance in a configuration. It thus seems easier to interpret the result of an asymmetric MDS analysis when both symmetric and asymmetric aspects of any proximity value as well as the proximity value itself are represented as distances in a configuration.

In the common space of Zielman's (1991) model, both symmetric and asymmetric proximity relationships are represented in one configuration. In a configuration of objects for each source, symmetric and asymmetric proximity relationships are represented in the same configuration, when one of the two alternatives of representing asymmetric relationship is adopted. Thus Zielman's (1991) model fulfills the characteristic (a). Zielman and Heiser's (1991, 1993) model fulfills the characteristic (a), but as mentioned earlier, the asymmetric proximity relationship is represented by projections (of object vectors on the slide vector) which correspond to the square of a proximity value. The model of DeSarbo et al. (1992) represents the symmetric proximity relationship common to all sources in a configuration of objects but does not represent the asymmetric proximity relationship in the configuration, and thus fails criterion (a).

Zielman's (1991) model fulfills the characteristic (b), because each source has its own configuration of objects to represent individual differences in symmetric proximity relationships, and sets of weights are used to represent individual differences in asymmetric proximity relationships. The model of DeSarbo et al. (1992) does not represent individual differences in symmetric and asymmetric proximity relationships geometrically but uses two different sets of parameters to represent two respective sorts of individual differences and thus fulfills the characteristic (b). Zielman and Heiser's (1991, 1993) model does not fulfill the characteristic (b), because one set of weights is used to represent individual differences both in symmetric and asymmetric proximity relationships.

Zielman and Heiser's (1991, 1993) model represents both a proximity value and its corresponding symmetric element as distances but does not represent the asymmetric element as a distance (which is instead represented as the differences among projections that correspond to the squares of proximities). Thus Zielman and Heiser's (1991, 1993) model only partially fulfills the characteristic (c). The common space of Zielman's (1991) model represents both symmetric and asymmetric elements of a proximity value as well as the value itself as distances. In a configuration of objects for each source, symmetric and asymmetric elements of a proximity value can be represented as distances, when one of the two alternatives to represent an asymmetric proximity relationship is used. But a proximity value itself is not represented as a distance even if that alternative is chosen. Therefore Zielman's (1991) model partially fulfills the characteristic (c). The model of DeSarbo et al. (1992) is based not on distance but instead on Tversky's (1977) "contrast model." Thus, no asymmetric MDS model for two-mode three-way proximities fulfills all three characteristics simultaneously. Relationships between the models and their characteristics are summarized in Table 1.

The purpose of the present paper is to develop an asymmetric MDS model and associated algorithm, to account for individual differences in asymmetric proximity relationships, which fulfills the characteristics (a) through (c) by analyzing two-mode three-way proximities, which consist of two or more square asymmetric proximity matrices. The model and the associated algorithm are extensions of Okada and Imaizumi's (1987) model for the asymmetric MDS of one-mode two-way proximities. The present extension inherits the characteristic that the asymmetry parameters are linked to objects. This characteristic, combined with (a) and (c), means we can very easily assess the asymmetry of each object pair visually by simply looking at the configuration of objects.

## 2. The Model

In both the present and predecessor models, each object is represented both by a point and a circle (sphere, hypersphere) centered at that point in a multidimensional Euclidean space. The configuration of objects is called the common object configuration, whose points represent symmetric proximity relationship among objects and whose circles represent asymmetric proximity relationship among objects for the 'group' of all sources.

Table 1

Summary Table of the Relationships

between the Models and Characteristics (a) through (c)

| Models | Characteristics | | |
|---|---|---|---|
| | (a) | (b) | (c) |
| Zielman (1991) | yes | yes | partially |
| Zielman & Heiser (1991, 1993) | yes | no | partially |
| DeSarbo et al. (1992) | no | yes | no |

(a) Symmetric and asymmetric proximity relationships are simultaneously represented in a configuration of objects.

(b) Individual differences in both symmetric and asymmetric proximity relationships are distinguished.

(c) The model is based on distance.

The present model inherited characteristics (a) and (c) from Okada and Imaizumi's (1987) work. Although that predecessor cannot deal with individual differences in both symmetric and asymmetric proximity relationships, the components respectively representing symmetric and asymmetric proximity relationships are distinguished and separated (Okada 1990), as they are in the present model. Each source has its own weight applied to a configuration of points and a set of weights applied to the radii along dimensions. A weight applied to the configuration of points, called the symmetry weight, represents individual differences among sources in symmetric proximity relationships, and a set of weights applied to the radii, called the asymmetry weight, represents individual differences among sources in asymmetric proximity relationships.

Each source has its own configuration of objects, where each is represented by a point and an ellipse (two-dimensional space), an ellipsoid (three-dimensional space) or a hyperellipsoid (four- or higher-dimensional space). The configuration of points in a common object configuration is stretched or shrunk by applying the symmetry weight of a source, to derive its configuration points for the given source. The ellipse (ellipsoid, hyperellipsoid) embedded in a configuration of objects is derived by applying the asymmetry weights of the respective source to the radius of the circle in the common object configuration.

Let $s_{jki}$ be the observed proximity from object $j$ to object $k$ for source $i$, $n$ be the number of objects, and $N$ be the number of sources. Two-mode three-way asymmetric proximities consist of $N$ square $n \times n$ asymmetric proximity matrices, one from each of the $N$ sources. The $(j,k)$-th element of the $i$-th proximity matrix is $s_{jki}$ where $s_{jki}$ is not necessarily equal to $s_{kji}$.

Each object is represented by a point and a circle (sphere, hypersphere) in a $p$-dimensional Euclidean space. Let $x_{jt}$ be the coordinate of the point representing object $j$ on dimension $t$ for the common object configuration, and $r_j$ be the radius of the circle (sphere, hypersphere) centered at the point representing object $j$. A common object configuration in a two-dimensional space is represented in Figure 1.

The distance between two points representing objects $j$ and $k$ in the common object configuration is given by

$$d_{jk} = [\sum_{t=1}^{p} (x_{jt} - x_{kt})^2]^{1/2} , \quad (2.1)$$

which represents the symmetric element of the proximity between objects $j$ and $k$ in the common object configuration. The latter can be interpreted similarly as with the configuration of objects of Okada and Imaizumi (1987) for one-mode two-way asymmetric MDS. Thus, an object with larger radius is more similar to an object with smaller radius than vice versa for the 'group' of all sources. In the common object configuration $-(r_j - r_k)$ and $-(r_k - r_j)$ represent the skew symmetric elements of the proximities respectively from object $j$ to $k$ and from object $k$ to $j$. The skew symmetric element can be interpreted as a dominance effect (Zielman and Heiser 1996). For example, larger $r_j$ means the relative vulnerability of brand $j$ in brand switching (Okada 1988a).

The distance between two points representing objects $j$ and $k$ in the configuration of objects for source $i$ is defined by

$$d_{jki} = w_i d_{jk} . \quad (2.2)$$

Symmetry weight $w_i (w_i \geq 0)$ stretches or shrinks the configuration of points for source $i$. Asymmetry weight $u_{it}$ stretches or shrinks the radius of a circle

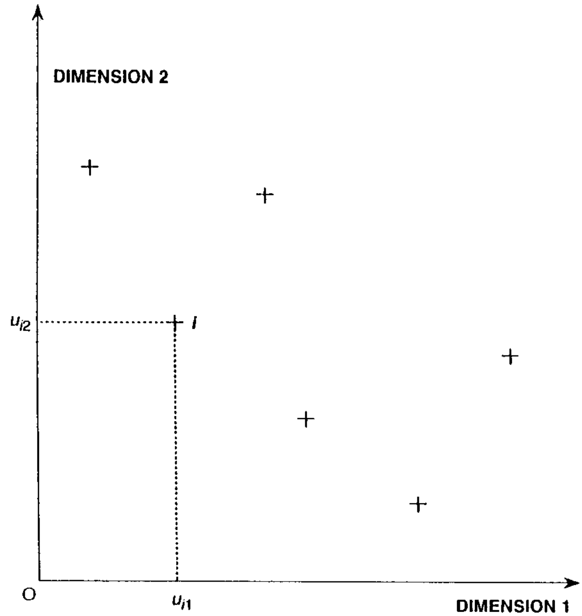


Figure 1. Common object configuration. Five objects are represented in a common object configuration. Each object is represented as a point (+) and a circle centered at that point in a two-dimensional space. An object shown at the bottom of the figure, represented only by a point, has a circle of radious = 0. Object $j$ is represented as a point $(x_{j1}, x_{j2})$ and a circle of radius $r_j$.

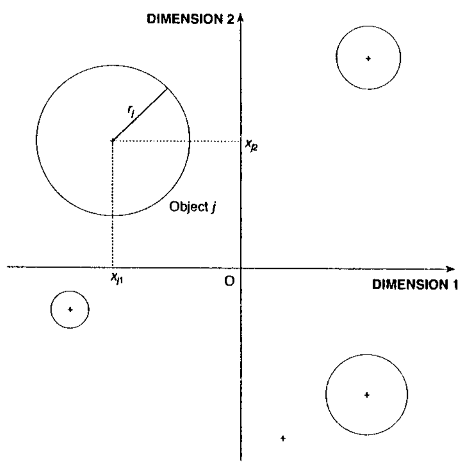


Figure 2. Asymmetry weight configuration. Six sources are represented in an asymmetry weight configuration. Each source is represented as a point (+) in a two-dimensional space. Source $i$ is represented as a point $(u_{i1}, u_{i2})$.

to provide the semiaxis along dimension $t$ of the ellipses in the configuration of objects for source $i$. The ellipse centered at the point representing object $j$ in the configuration of objects for source $i$ has semiaxes of length $u_{it}r_j$ along dimension $t$ ($t = 1, \ldots, p$), where $u_{it}$ ($u_{it} \geq 0$) is the asymmetry weight given to radius of the circle in the common object configuration.

Symmetry weight $w_i$ represents the salience or importance of the symmetric component for source $i$. Asymmetry weight $u_{it}$ represents the salience or importance of the asymmetric component for source $i$ along dimension $t$. Individual differences only in magnitude of symmetric relationships are represented without respect to separate dimensions. In contrast, individual differences in magnitude of asymmetric proximity relationships are represented along each dimension for each source. The present model focuses its attention more on individual differences in asymmetric proximity relationships than on individual differences in symmetric proximity relationships. Zielman and Heiser's (1991, 1993) and Zielman's model (1991) all represent individual differences in the magnitude of symmetric proximity relationships along each dimension, while the model of DeSarbo et al. (1992) represents individual differences in magnitude of symmetric proximity relationships. The present model adopts the same stance as that of DeSarbo et al. (1992) for representing individual differences in symmetric proximity relationships. Asymmetry weights are represented geometrically in the "asymmetry weight" configuration, where each source is represented as a point in a $p$-dimensional space. Figure 2 shows an asymmetry weight configuration in a two-dimensional space.

It is assumed that for source $i$, $s_{jki}$ is monotonically decreasingly (when the observed $s_{jki}$ depicts similarity) or increasingly (when $s_{jki}$ depicts dissimilarity) related to $m_{jki}$, where $m_{jki}$ is defined as

$$m_{jki} = d_{jki} - v_{jki}\, r_j + v_{kji}\, r_k \,. \tag{2.3}$$

$v_{jki}\, r_j$ and $v_{kji}\, r_k$ are introduced to account for skew symmetries in $s_{jki}$, and $v_{jki}$ is defined as

$$v_{jki} = z_{jki}^{1/2} \,, \tag{2.4}$$

where

$$z_{jki} = \frac{d_{jki}^2}{\sum_{t=1}^{p} \left(\frac{x_{jt} - x_{kt}}{u_{it}}\right)^2} \,. \tag{2.5}$$

$v_{kji}$ and $z_{kji}$ are defined simply by interchanging $j$ and $k$ in Equations (2.4) and (2.5), and are equal to $v_{jki}$ and $z_{jki}$ respectively. The two-way model of Okada and Imaizumi (1987) is the case where subscript $i$ is eliminated and

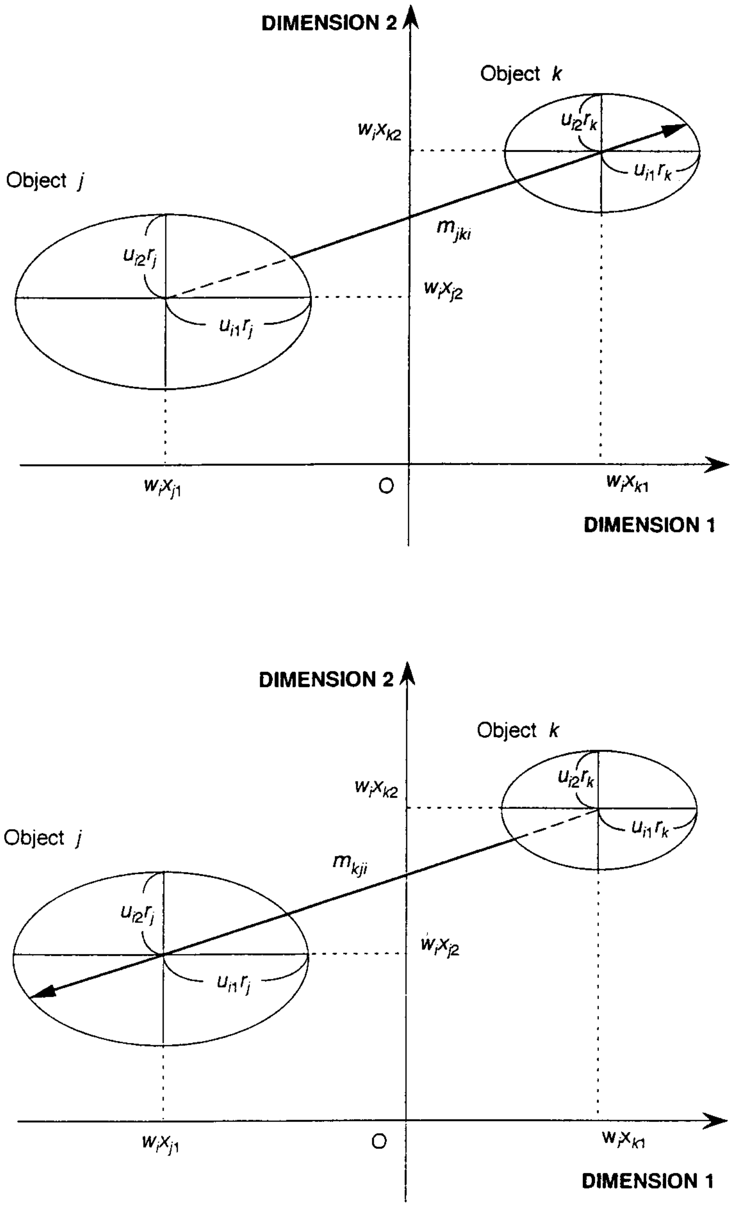


Figure 3. $m_{jki}$ and $m_{kji}$ in the configuration of objects for source $i$.

$v_{jki} = v_{kji} = 1$ in Equation (2.3). In the two-way model of Okada and Imaizumi (1987) the asymmetry is unidimensional, whereas in the present model asymmetry of each source is multidimensional.

In Figure 3, $j$ and $k$ are represented in a two-dimensional configuration of objects for source $i$. Object $j$ is represented as a point whose coordinates are $(w_i x_{j1}, w_i x_{j2})$ and an ellipse, centered at that point, which has semiaxes of length $u_{i1} r_j$ along dimension 1 and $u_{i2} r_j$ along dimension 2. Object $k$ is represented both as a point whose coordinates are $(w_i x_{k1}, w_i x_{k2})$ and as an ellipse, centered at that point, which has semiaxes of length $u_{i1} r_k$ and $u_{i2} r_k$. The lengths of the solid arrows in the upper and lower panels respectively depict $m_{jki}$ and $m_{kji}$. The distance between the two points representing objects $j$ and $k$ depicts $d_{jki}$. The lengths of the dashed line in the ellipse representing object $j$ in the upper panel and object $k$ in the lower panel respectively depict $v_{jki} r_j$ and $v_{kji} r_k$.

In the present model, both symmetric and asymmetric proximity relationships are represented in the same configuration, the former as interpoint distances, and the latter as differences of radii of circles in the common object configuration or as differences of two distances between the ellipses (shown as lengths of the dashed line in Figure 3) in the configuration of objects for each source. Two components representing symmetric and asymmetric proximity relationships are clearly distinguished and separated (see Equation (3.5)). The proximity itself as well as symmetric and asymmetric elements of the proximity are represented as distances in the configuration. Each object has its own circle or ellipse to account for asymmetry. Thus, the characteristics (a) through (c) which characterize the development of the present asymmetric MDS model mentioned are satisfied.

In the predecessor model for one-mode two-way asymmetric proximities, the orientation of the dimensions of a configuration is determined uniquely up to orthogonal rotations (Okada and Imaizumi 1987). In the present model, the orientation of the dimensions of a common object configuration is determined uniquely up to reflections and the permutations of the axes because of the introduction of the asymmetry weights. When the orientation of the dimensions is arbitrarily altered, the asymmetry weights are applied to differently oriented dimensions, and badness-of-fit consequently increases. This aspect of the model leads to the uniquely oriented dimensions of the common object configuration.

## 3. The Algorithm

A nonmetric algorithm to derive the common object configuration ($x_{jt}$; $j = 1, \ldots, n$, $t = 1, \ldots, p$ and $r_j$; $j = 1, \ldots, n$) and asymmetry weights ($u_{it}$; $i = 1, \ldots, N$, $t = 1, \ldots, p$) by analyzing two-mode three-way asymmetric proximities $s_{jki}$ $(j,k[j \neq k] = 1, \ldots, n)$ among $n$ objects from $N$ sources will be presented. Symmetry weights ($w_i$; $i = 1, \ldots, N$), dependent on **U**, **X** and **r**, are derived through the normalization of $m_{jki}$ (see Equation (3.7)). The algorithm is an extension of the predecessor model for one-mode two-way asymmetric proximities (Okada and Imaizumi 1987).

The measure of badness-of-fit between $m_{jki}$ and the monotone transformed $s_{jki}$, called stress $S$, is based on the Stress Formula 2 (Kruskal and Carroll 1969) and defined as

$$S = \left[ \frac{1}{N} \sum_{i=1}^{N} S_i^2 \right]^{1/2}, \tag{3.1}$$

where $S_i$ is the measure of badness-of-fit for source $i$, and is defined as

$$S_i = \left[ \frac{\sum_{j=1}^{n} \sum_{\substack{k=1 \\ j \neq k}}^{n} (m_{jki} - \hat{m}_{jki})^2}{\sum_{j=1}^{n} \sum_{\substack{k=1 \\ j \neq k}}^{n} (m_{jki} - \bar{m}_i)^2} \right]^{1/2} \tag{3.2}$$

In equation (3.2) $\hat{m}_{jki}$ is the transformed $s_{jki}$ defined by Kruskal's (1964b) monotone algorithm, and $\bar{m}_i$ is

$$\bar{m}_i = \frac{\sum_{j=1}^{n} \sum_{\substack{k=1 \\ j \neq k}}^{n} m_{jki}}{n(n-1)} \tag{3.3}$$

which represents the mean of $m_{jki}$ for source $i$. The stress $S$ is a function of $x_{jt}$, $r_j$, and $u_{it}$. The problem here is to find $x_{jt}$, $r_j$, and $u_{it}$ (as well as $w_i$) which minimize $S$ in a Euclidean space of a given dimensionality. The iterative algorithm to minimize $S$ is based on the steepest descent method.

### 3.1 Summary of the Algorithm

As illustrated in Figure 4, the algorithm has three segments. The first is reading proximity data and constructing initial configurations and values. The second is the iterative process which minimizes $S$ in a Euclidean space of a given dimensionality $p$, and the third is standardizing the obtained result. After constructing initial configurations and values of the common object configuration $\mathbf{X} = [x_{jt}]$ and $\mathbf{r} = [r_j]$, the asymmetry weight configuration $\mathbf{U} = [u_{it}]$ and symmetry weights $\mathbf{w} = [w_i]$, the iterative process begins. It consists of (a) updating radii $\mathbf{r} = [r_j]$, for existing **X**, **U**, and **w**, (b) updating **X**, (c) normalizing **X** and $m_{jki}$, and deriving **w** through the normalization, (d) updating **U**, and (e) normalizing **X** and $m_{jki}$, and deriving **w** through the normalization. At each iteration, $S$ is calculated to check for convergence. If not, a new iteration begins. If no further iterations are needed, the iteration is terminated and the standardization of **X**, **U**, and **r**, as well as **w** is executed. When $S$ becomes small enough to be neglected ($S \leq 0.1\text{E-}4$) or the difference between $S$ of the present iteration and that of the previous iteration becomes negligible ($|\text{difference}| \leq 0.1\text{E-}7$), it is judged that $S$ converged.

### 3.2 Initial Configurations and Initial Values

Constructing initial configurations and initial values for **X**, **r**, **U**, and **w** depends on whether a higher dimensional result of the analysis of the same data is available. If so, initial configurations and initial values can be derived from it; otherwise, they are derived from the data.

When the higher dimensional result is not available, the initial (configuration of points of the) common object configuration **X** is derived from the observed proximities. Each of $N$ proximity matrices are symmetrized by averaging the conjugate elements. For each proximity matrix, an additive constant is calculated (Torgerson 1952) and added to all elements of that proximity matrix. Each matrix is then converted to a scalar product matrix by double centering the matrix, and is normalized so that sum of squares of all elements of the matrix is equal to 1. The mean matrix of $N$ scalar product matrices is calculated. A $p$-dimensional configuration of objects is derived from the mean scalar product matrix by Torgerson's (1952) method. The derived configuration of objects is used as an initial common object configuration **X**. The initial asymmetry weight configuration consists of $N \times p$ 1's (all $u_{it} = 1$). We use 1 as the initial value of all symmetry weights $w_i$ ($i = 1, \ldots, N$), and 0 as the initial value of all radii $r_j$ ($j = 1, \ldots, n$).

When the higher dimensional result is available, radii and symmetry weights of the higher dimensional results are used as their initial values respectively. The initial common object configuration **X** and the initial asymmetry weight configuration are derived by extracting $p$ most "heavily contributing" dimensions from the higher dimensional result. The "contribution" of dimension $t$ is based on the sum of two elements: the sum of squares of $x_{jt}$ ($j = 1, \ldots, n$) which reflects the salience of the symmetric component of the results, and the sum of squares of $u_{it}$ ($i = 1, \ldots, N$) which reflects the salience of the asymmetric component of the results. The details of defining the "contribution" will be described in Section 3.7.

Another option for deriving initial common object configuration **X** and radii **r** is to use the resultant configuration of objects and radii of the

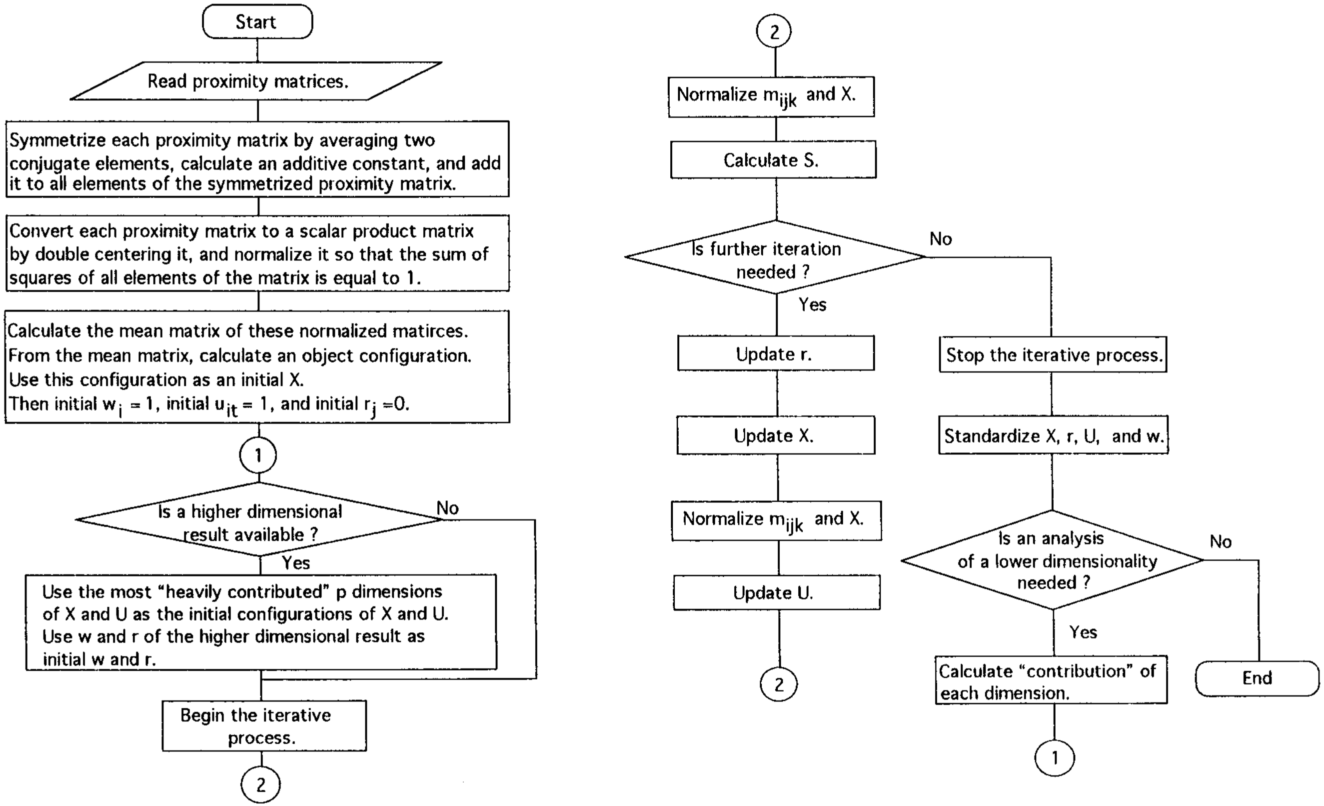


Figure 4. Flow of the algorithm.

asymmetric MDS for one-mode two-way proximities (Okada and Imaizumi 1987). Two-mode three-way proximities can be analyzed by that asymmetric MDS procedure as $N$ replications of $n \times n$ proximity matrix.

### 3.3 Normalization

After constructing initial configurations and values, the iterative process begins. At the beginning of each new iteration, as well as after updating the common object configuration **X** and the asymmetry weight configuration **U**, the common object configuration **X** is normalized so that

$$\sum_{j=1}^{n} x_{jt} = 0 \ \ (\text{for } t = 1, \ldots, p), \quad \text{and} \quad \sum_{j=1}^{n} \sum_{t=1}^{p} x_{jt}^2 = n \,. \tag{3.4}$$

The asymmetry weight configuration **U** is uniformly stretched or shrunk by multiplying the scalar used in the normalization of the right formula of Equation (3.4).

As shown in Okada (1990) the sum of squared deviations of the $m_{jki}$ is additively decomposed into two terms:

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \sum_{k=1}^{n} (m_{jki} - \overline{m}_i)^2 = n(n-1)\, V_i(d_{jki}) + n(n-1)\, V_i \left[ \frac{m_{jki} - m_{kji}}{2} \right], \tag{3.5}$$

where $V_i(d_{jki})$ and $V_i[(m_{jki} - m_{kji})/2]$ represent the variance of $d_{jki}$ and of $(m_{jki} - m_{kji})/2$ for source $i$ respectively. The first and the second terms of the right side of Equation (3.5) represent the magnitude of symmetric and asymmetric components of the configuration of objects for source $i$ (Okada, 1990). Because $V_i(d_{jki})$ is equal to $w_i^2\, V(d_{jk})$, Equation (3.5) can be rewritten as

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \sum_{k=1}^{n} (m_{jki} - \overline{m}_i)^2 = n(n-1)[w_i^2\, V(d_{jk}) + V_i \left[ \frac{m_{jki} - m_{kji}}{2} \right]], \tag{3.6}$$

where $V(d_{jk})$ represents the variance of $d_{jk}$. For source $i$ $m_{jki}$ is normalized by applying a scalar multiple to $u_{it}$ ($t = 1, \ldots, p$) and to $w_i$ so that

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \sum_{k=1}^{n} (m_{jki} - \overline{m}_i)^2 = n \,, \tag{3.7}$$

i. e., the sum of symmetric and asymmetric components is constant for all sources.

As will be noted later in Section 3.5, only **U** is improved in the iteration, while **w** is unchanged from the previous iteration before the normalization of $m_{jki}$ shown by Equation (3.7). The normalization of $m_{jki}$ is effected by multiplying $u_{it}$ $(t = 1, \dots, p)$ and to $w_i$ by a scalar to stretch or shrink them uniformly. Because $u_{it}$ $(t = 1, \dots, p)$ were updated from the previous iteration, $w_i$ is updated for the current $u_{it}$ $(t = 1, \dots, p)$ by the normalization.

### 3.4 Updating Radii

Radii are updated for the current **X**, **U**, and **w** by obtaining a least squares solution which minimize the criterion $LS_r$

$$LS_r = \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ j \neq k}}^{n} \left[ \frac{(\hat{m}_{jki} - d_{jki})}{v_{jki}} - (r_j - r_k) \right]^2 , \tag{3.8}$$

and the obtained radii are normalized so that

$$\min_j r_j = 0 . \tag{3.9}$$

Thus **r**, which minimizes $LS_r$ for the current **X**, **U**, and **w**, where **X**, **U**, and **w** minimize $S$, is derived, and **r** which directly minimizes $S$ is not derived because of the difficulty in the computations.

### 3.5 Updating the Common Object Configuration X and the Asymmetry Weight Configuration

The common object configuration **X** and the asymmetry weight configuration are updated by the steepest descent method, where the step size is calculated by the linear search method which evaluates $S$ at $\alpha = 0.0$, 0.1 and 0.2 of the equation corresponding Kruskal's (1964b, p. 120), where the partial derivatives of $S$ with respect to $x_{jt}$ and $u_{it}$ to obtain negative gradients are presented. The partial derivative of $S$ with respect to $y$, where $y$ represents either $x_{jt}$ or $u_{it}$ , is

$$\frac{\partial S}{\partial y} = \frac{\partial S}{\partial S_i} \frac{\partial S_i}{\partial y} = \frac{\partial S}{\partial S_i} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ j \neq k}}^{n} \left[ \frac{S_i}{S_i^*} (m_{jki} - \hat{m}_{jki}) - \frac{S_i}{T_i^*} (m_{jki} - \bar{m}_i) \right] \frac{\partial m_{jki}}{\partial y} \tag{3.10}$$

where

$$S_i^* = \sum_{j=1}^{n} \sum_{\substack{k=1 \\ j \neq k}}^{n} (m_{jki} - \hat{m}_{jki})^2 , \tag{3.11}$$

and

$$T_i^* = \sum_{\substack{j=1\\j\neq k}}^{n} \sum_{k=1}^{n} (m_{jki} - \overline{m}_i)^2 . \tag{3.12}$$

In Equation (3.10) $\dfrac{\partial m_{jki}}{\partial y}$ represents $\dfrac{\partial m_{jki}}{\partial x_{lt}}$ or $\dfrac{\partial m_{jki}}{\partial u_{ht}}$. $\dfrac{\partial m_{jki}}{\partial x_{lt}}$ is given by Equation (3.13), and $\dfrac{\partial m_{jki}}{\partial u_{ht}}$ is given by Equation (3.18). $\dfrac{\partial m_{jki}}{\partial x_{lt}}$ is

$$\frac{\partial m_{jki}}{\partial x_{lt}} = \frac{\partial d_{jki}}{\partial x_{lt}} - (r_j - r_k) \frac{\partial v_{jki}}{\partial x_{lt}} , \tag{3.13}$$

where

$$\frac{\partial d_{jki}}{\partial x_{lt}} = \frac{w_i (x_{jt} - x_{kt})(\delta^{jl} - \delta^{kl})}{d_{jki}} , \tag{3.14}$$

and

$$\frac{\partial v_{jki}}{\partial x_{lt}} = \left[ \frac{w_i^2 (x_{js} - x_{ks})}{V_{Tjki}^*} - \frac{x_{js} - x_{ks}}{u_{iq}^2 \, V_{Sjki}^*} \right] (\delta^{jl} \, \delta^{ts} - \delta^{kl} \, \delta^{ts}) . \tag{3.15}$$

$V_{Sjki}^*$ and $V_{Tjki}^*$ in Equation (3.15) are given by

$$V_{Sjki}^* = w_i^2 \sum_{t=1}^{p} (x_{jt} - x_{kt})^2 , \tag{3.16}$$

and

$$V_{Tjki}^* = \sum_{t=1}^{p} \left( \frac{x_{jt} - x_{kt}}{u_{it}} \right)^2 \tag{3.17}$$

respectively. $\dfrac{\partial m_{jki}}{\partial u_{ht}}$ is

$$\frac{\partial m_{jki}}{\partial u_{ht}} = - (r_j - r_k) \frac{\partial v_{jki}}{\partial u_{lt}} , \tag{3.18}$$

where

$$\frac{\partial v_{jki}}{\partial u_{lt}} = \frac{w_i \, d_{jki}}{v_{jki} \, u_{is}^3 \sum_{s'=1}^{p} \left( \dfrac{x_{js'} - x_{ks'}}{u_{is'}} \right)^2} \, \delta^{il} \, \delta^{ts} . \tag{3.19}$$

### 3.6 Standardization of the Resultant Configurations

The resultant **X**, **r**, **U**, and **w** have an indeterminacy. When **X** and **r** are multiplied by a constant, the effect of the constant can be canceled by multiplying **U** and **w** by its reciprocal. As shown in Equation (3.5), for source $i$, the sum of squared deviation of $m_{jki}$ from the mean $\bar{m}_i$ is decomposed into two terms, where the first and the second terms of the right side respectively represent the magnitude of symmetric and asymmetric components of the configuration of objects for source $i$. The similar decomposition is possible for a common object configuration

$$\sum_{\substack{j=1 \\ j\neq k}}^{n}\sum_{k=1}^{n} (m_{jk} - \bar{m})^2 = n(n-1)\,V(d_{jk}) + n(n-1)\,V\left[\frac{m_{jk} - m_{kj}}{2}\right]$$

$$= n(n-1)\,V(d_{jk}) + n(n-1)\,V(r_j - r_k)\,, \qquad (3.20)$$

where

$$m_{jk} = d_{jk} - r_j + r_k\,, \qquad (3.21)$$

and

$$\bar{m} = \frac{\sum_{\substack{j=1 \\ j\neq k}}^{n}\sum_{k=1}^{n} m_{jk}}{n(n-1)}\,. \qquad (3.22)$$

A common object configuration is standardized so that the first and the second terms of the right side of Equation (3.20) respectively reflect the average relative magnitudes of symmetric and asymmetric components in $N$ configurations of objects. The average relative magnitude of symmetric component $q_s$ over the $N$ sources and the average relative magnitude of asymmetric component $q_a$ over the $N$ sources are defined by

$$q_s = \frac{\sum_{i=1}^{N} w_i^2\, V(d_{jk})}{\sum_{i=1}^{N} [w_i^2\, V(d_{jk}) + V_i(r_j - r_k)]}\,, \qquad (3.23)$$

and

$$q_a = \frac{\sum_{i=1}^{N} V_i(r_j - r_k)}{\sum_{i=1}^{N} [w_i^2 V(d_{jk}) + V_i(r_j - r_k)]} , \tag{3.24}$$

respectively. The common object configuration **X** is already normalized to satisfy Equation (3.4); thus radius **r** is standardized by a multiplicative constant so that

$$\frac{V(d_{jk})}{V(r_j - r_k)} = \frac{q_s}{q_a} . \tag{3.25}$$

**U** is multiplied by the reciprocal of the constant. This standardization enables the common object configuration to depict the magnitude of symmetric and asymmetric components in a manner similar to the configuration of objects in Okada and Imaizumi (1987).

### 3.7 Remarks on the Algorithm

The present algorithm allows for missing data simply by omitting them (Kruskal 1964a) in Equations (3.2), (3.3), (3.10), (3.11), and (3.12). The asymmetry weights $u_{it}$ and symmetry weights $w_i$ are assumed to be nonnegative in the present model. The former are kept nonnegative by changing signs whenever an updated value becomes negative. Thanks to Equation (2.5), $z_{jki}$ is not affected by the sign of $u_{it}$. As $w_i$ is derived through the normalization by multiplying a scalar, it is kept nonnegative provided that the initial $w_i$ is nonnegative.

When the higher dimensional result is available, both the initial common object configuration **X** and the initial asymmetry weight configuration are derived by extracting the $p$ most "heavily contributing" dimensions from the higher dimensional result. The "contribution" of dimension $t$ of the standardized **X** and **U** is defined by the sum of (a) the sum of squares of $x_{jt}$ ($j = 1, \ldots, n$) which reflects the average relative salience of symmetric component and (b) the sum of squares of $u_{it}$ ($i = 1, \ldots, N$) which reflects the average relative salience of the asymmetric component; i.e., the contribution of dimension $t$ is defined by

$$b_t = \sum_{j=1}^{n} x_{jt}^2 + \sum_{i=1}^{N} u_{it}^2 . \tag{3.26}$$

The initial common object configuration **X** and the asymmetric weight configuration are constructed from the $p$ dimensions which have largest $b_t$ , where $p$ is the dimensionality specified for the analysis.

### 3.8 Procedure to Find the Solution

The procedure to analyze two-mode three-way asymmetric proximities by the present algorithm consists of (a) determining the maximum dimensionality to be used in the analysis, (b) analyzing the proximities by the present algorithm in the spaces from the maximum dimensionality through unidimensionality to obtain a solution in each of the dimensionalities, and (c) selecting the ''best'' result as the solution. The selection of the solution over the different dimensionalities is based on the elbow criterion of the minimized $S$ and on the interpretation of the results (Okada and Imaizumi 1987). The procedure mentioned above relies on rational initial configurations and values. We alternatively use a random initial common object configuration.

## 4. An Application

The present model and its algorithm were used to analyze intergenerational occupational mobility among eight occupational categories in four years in Japan (Seiyama, Naoi, Sato, Tsuzuki and Kojima 1990, pp. 46-47, Table 2.12). The data consist of four $8 \times 8$ transition tables. The $(j,k)$-th element of each table represents the number of sons whose occupations are in occupational category $k$ and whose fathers' occupations are (were) in occupational category $j$. Thus, the $(j,k)$-th element of the $i$-th table represents the number of intergenerational occupational movements from the fathers' occupational category $j$ to the sons' occupational category $k$ in year $i$. The four tables correspond to data from 1955, 1965, 1975, and 1985. The eight occupational categories are

(1) **Professional** occupations,
(2) **Non-manual** occupations employed by **large** enterprises,
(3) **Non-manual** occupations employed by **small** enterprises,
(4) **Non-manual self**-employed occupations,
(5) **Manual** occupations employed by **large** enterprises,
(6) **Manual** occupations employed by **small** enterprises,
(7) **Manual self**-employed occupations, and
(8) **Farm** occupations.

These four tables constitute two-mode three-way proximities. The emboldened word(s) above are used to represent each occupational category in Figure 5.

In each of the four tables, there are differences among eight row and column marginals which represent the differences in shares of manpower among occupational categories. Stated differently, there are differences in the sum of inflow and outflow to/from each of the eight occupational categories.

Table 2
Rescaled Intergenerational Occupational Mobility Table

| Father's occupation | Son's occupation 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | Year 1955 | | | | | | | |
| 1 | 118.2 | 33.8 | 39.3 | 3.4 | 3.1 | 8.6 | 9.1 | 5.7 |
| 2 | 39.5 | 53.7 | 39.4 | 22.0 | 15.7 | 17.3 | 7.6 | 10.8 |
| 3 | 4.9 | 29.5 | 42.9 | 23.6 | 16.5 | 30.2 | 7.9 | 0.0 |
| 4 | 35.5 | 50.8 | 59.0 | 73.0 | 20.7 | 22.5 | 23.6 | 5.3 |
| 5 | 9.4 | 9.4 | 54.9 | 5.7 | 73.7 | 38.6 | 11.8 | 3.5 |
| 6 | 5.8 | 14.4 | 15.1 | 10.4 | 45.0 | 64.9 | 20.2 | 7.1 |
| 7 | 13.6 | 36.4 | 37.0 | 16.4 | 42.3 | 65.2 | 71.8 | 9.2 |
| 8 | 23.5 | 37.5 | 28.8 | 27.0 | 47.5 | 41.6 | 27.7 | 98.2 |
| | Year 1965 | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 102.9 | 32.9 | 26.4 | 3.9 | 14.2 | 19.8 | 11.6 | 1.6 |
| 2 | 45.3 | 50.7 | 31.7 | 18.3 | 22.2 | 13.8 | 9.7 | 4.4 |
| 3 | 22.6 | 23.2 | 54.1 | 13.8 | 26.2 | 17.7 | 5.9 | 0.8 |
| 4 | 28.9 | 44.2 | 41.5 | 94.9 | 20.8 | 25.9 | 17.2 | 3.7 |
| 5 | 17.0 | 31.8 | 34.9 | 1.5 | 56.9 | 42.0 | 10.4 | 3.1 |
| 6 | 2.5 | 12.5 | 33.0 | 6.5 | 34.4 | 53.2 | 15.6 | 5.9 |
| 7 | 7.7 | 42.1 | 43.5 | 30.3 | 37.2 | 50.6 | 78.9 | 5.0 |
| 8 | 34.9 | 41.3 | 45.5 | 28.7 | 65.5 | 88.5 | 30.4 | 57.9 |
| | Year 1975 | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 123.3 | 44.3 | 52.8 | 15.6 | 13.5 | 11.0 | 7.4 | 3.7 |
| 2 | 37.5 | 76.6 | 46.4 | 27.5 | 24.7 | 22.2 | 11.2 | 4.1 |
| 3 | 14.7 | 30.3 | 49.1 | 11.4 | 17.0 | 34.4 | 1.9 | 2.4 |
| 4 | 34.3 | 55.0 | 71.9 | 89.5 | 36.2 | 32.6 | 38.9 | 3.3 |
| 5 | 29.8 | 29.6 | 34.0 | 12.1 | 75.8 | 54.7 | 10.7 | 3.6 |
| 6 | 12.8 | 16.7 | 36.3 | 7.1 | 40.6 | 66.7 | 12.0 | 2.4 |
| 7 | 27.6 | 36.9 | 55.8 | 28.7 | 40.8 | 52.8 | 106.1 | 6.9 |
| 8 | 33.0 | 45.1 | 76.9 | 30.9 | 85.5 | 115.4 | 40.8 | 65.2 |
| | Year 1985 | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 88.4 | 37.4 | 49.0 | 7.4 | 6.7 | 11.5 | 1.5 | 0.0 |
| 2 | 39.9 | 63.0 | 42.8 | 17.2 | 24.8 | 14.8 | 7.1 | 1.6 |
| 3 | 11.1 | 23.4 | 47.2 | 8.9 | 11.3 | 28.2 | 6.2 | 0.0 |
| 4 | 39.8 | 32.6 | 58.0 | 97.4 | 25.4 | 22.1 | 18.4 | 2.4 |
| 5 | 42.5 | 32.6 | 36.1 | 4.5 | 72.6 | 27.1 | 12.4 | 0.0 |
| 6 | 14.0 | 23.7 | 38.5 | 7.6 | 32.2 | 52.7 | 15.9 | 3.6 |
| 7 | 18.4 | 34.9 | 37.1 | 24.6 | 31.1 | 57.4 | 95.8 | 3.8 |
| 8 | 44.3 | 41.6 | 55.6 | 36.8 | 68.7 | 98.4 | 40.2 | 51.8 |

The sums respectively reflect the degree of attractiveness of the occupational category which draws in manpower from other occupational categories and relegation of manpower to other occupational categories. It seems preferable to remove the differences in shares of manpower to disclose the structure of intergenerational occupational mobility among the eight occupational categories (Harshman et al. 1982; Okada 1988b; Slater 1976).

The $i$-th table was rescaled by multiplying with a rescaling constant $c_{ij}$ applied to row $j$ and column $j$ so that, for $j = 1, \ldots ,8$, the sum of row $j$ plus column $j$ elements of the rescaled table is equal to the mean sum of row plus column elements of the raw $i$-th table (Harshman et al. 1982, p. 229; Okada 1988a, 1988b). The rescaled table is represented in Table 2.

The two-mode, three-way set of four rescaled intergenerational occupational mobility tables was analyzed by the present asymmetric MDS. The analysis with rational initial configurations and values was done using the maximum dimensionalities of eight through four. Then one $S$ was obtained in eight-dimensional space, two $S$ in seven-dimensional space, three $S$ in six-dimensional space, ..., and five $S$ in four- through unidimensional spaces. The smallest $S$ in each dimensional space was chosen as the minimized $S$ in that dimensional space. The resulting minimized $S$ in five- through unidimensional spaces were 0.335, 0.335, 0.349, 0.378, and 0.435. The minimized $S$ seem to indicate that the two- dimensional result is appropriate to be a solution. To obtain a baseline for comparison of this preferred solution, we conducted analyses using 100 different initial common object configurations $\mathbf{X}$ generated from uniformly distributed random numbers with $r_j = 0$, $u_{it} = 1$ and $w_i = 1$ as initial radii, an initial asymmetry weight configuration and initial values of symmetry weights in a two-dimensional space. All results given by these analyses yielded $S$ values which were larger than 0.378 obtained above. The smallest $S$ among 100 results obtained was 0.395. Thirty eight of 100 resultant $S$ values were smaller than 0.5. The set of four rescaled tables was analyzed by Okada and Imaizumi's (1987) one-mode two-way asymmetric MDS as four replications of an $8 \times 8$ similarity matrix. Then the two-mode three-way analysis using the obtained two-dimensional configuration of objects and radii as the initial common object configuration for $\mathbf{X}$ and $\mathbf{r}$ with $u_{it} = 1$ and $w_i = 1$ was conducted. The analysis yielded as $S$ value of 0.384. Thus, the two-dimensional result having an $S$ value of 0.378 should be chosen as the best solution. The two-dimensional common object configuration is illustrated in Figure 5. Symmetry weights $w_i$ ($i = 1, \ldots ,4$) and asymmetry weights $u_{it}$ ($i = 1, \ldots ,4$; $t = 1,2$) are shown in Table 3. The asymmetry weight configuration is illustrated in Figure 6.

In Figure 5 each occupational category is represented by a point and a circle in a two-dimensional configuration. In Figure 6, each year is represented by a point in a two-dimensional configuration. As noted earlier,

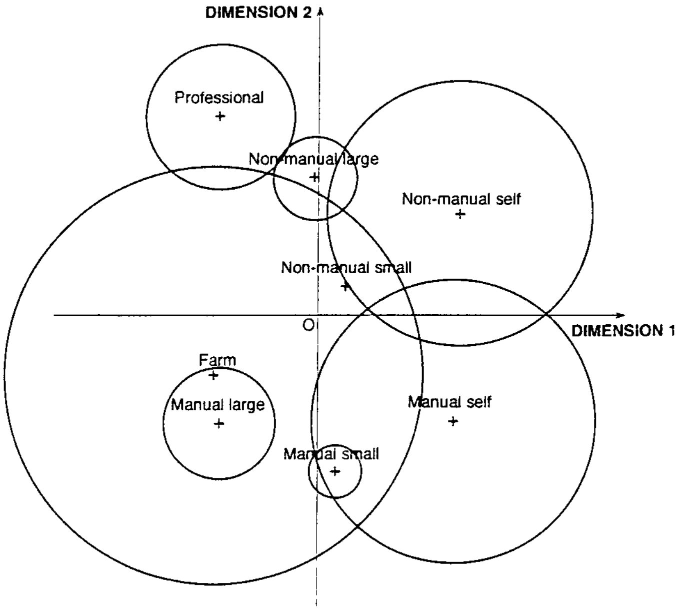


Figure 5. Common object configuration of eight occupational categories.

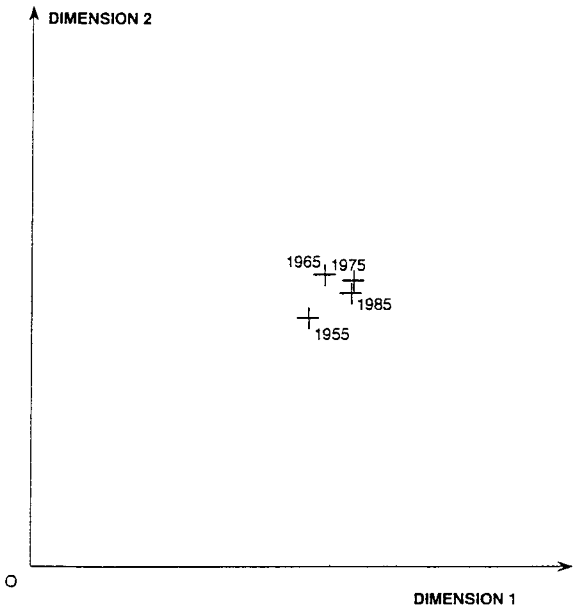


Figure 6. Asymmetry weight configuration of four years.

Table 3

Symmetry weight and Asymmetry weight

| Year | Symmetry weight | Asymmetry weight | |
|---|---|---|---|
| | | Dim 1 | Dim 2 |
| 1955 | 0.503 | 0.424 | 0.379 |
| 1965 | 0.428 | 0.454 | 0.446 |
| 1975 | 0.402 | 0.497 | 0.438 |
| 1985 | 0.427 | 0.496 | 0.415 |

the orientation of dimensions in a common object configuration is uniquely determined up to reflections and permutations and in an asymmetry weight configuration up to permutations. The orientation of the dimensions in Figures 5 and 6 is given by the analysis, and no rotation was applied to the obtained configurations. But similar to the INDSCAL model (Arabie, Carroll, and DeSarbo 1987, p. 25), when we have asymmetry weights on a diagonal (45 degree) line passing through the origin in any two-dimensional (sub)space of the asymmetry weight configuration, the orientation of the dimensions has an indeterminacy in that two-dimensional space. Although the asymmetry weights obtained are not exactly on a diagonal line, four points in Figure 6 seem to be close to the diagonal line. Therefore, the orientation of the dimensions might have indeterminacy. But as shown below, the dimensions have a very clear-cut interpretation, which suggests that no rotation is needed to interpret them. The uniqueness of the orientation of the dimensions will be further discussed below.

In the upper half of Figure 5, there are four non-manual occupational categories, and in the lower half of Figure 5 there are four manual occupational categories. Vertical Dimension 2 of the solution seems to represent the difference between non-manual and manual occupational categories. Horizontal Dimension 1 seems to differentiate among the self-employed, employees of small enterprises, employees of large enterprises, and professional or farm occupations.

In the present application, an occupational category with a larger radius means that sons whose fathers are (were) in the corresponding occupational category have a larger tendency of moving from their fathers' occupational category and that sons whose fathers are (were) in other occupational categories have a lesser tendency of moving into that occupational category. The resulting radius of farm occupations is the largest. This result is compatible with the fact that the number of farm laborers has drastically decreased (Seiyama et al. 1990, p. 20, p. 48). Self-employed occupational categories have larger radii than employed occupational categories. This result is also compatible with the movement of manpower from self-employed to employed occupational categories (Seiyama et al. 1990, pp. 43-44). Occupational categories along the periphery of the horizontal dimension have larger radii than in its center. Thus, manpower has been migrating from occupational categories in the periphery to those in the center of the horizontal dimension.

Symmetry weights decreased from 1955 to 1975, and increased in 1985, showing that the symmetric occupational mobility among eight occupational categories decreased from 1955 to 1975 and increased in 1985. Asymmetry weights along Dimension 1 were larger than those along Dimension 2 for each of the four years, showing that asymmetric occupational mobility along Dimension 1 is greater than that along Dimension 2, and suggesting that asymmetric occupational mobility within both non-manual and manual occupational categories was larger than that between non-manual and manual occupational categories. Asymmetry weights along Dimension 1 increased from 1955 to 1975 and showed a small decrease for 1985, suggesting that asymmetric occupational mobility within both non-manual and manual occupational categories increased from 1955 to 1975 and slightly decreased in 1985. The asymmetry weight along Dimension 2 in 1955 was distinctly smaller than those for 1965, 1975 and 1985, and those in 1965 and 1975 were almost constant, while that in 1985 was slightly smaller than those in 1965 and 1975, suggesting the asymmetric occupational mobility between non-manual and manual occupational categories increased in 1965 compared to 1955, was almost the same in 1975 as in 1965, and in 1985 was slightly less than in 1965 and 1975. This result is compatible with the behavior of the mobility between non-manual and manual occupational categories described in Seiyama et al. (1990, p. 31-32).

From 1955 to 1965, asymmetric occupational mobility both dimensions increased. The asymmetric occupational mobility along Dimension 2 increased more than the asymmetric occupational mobility along Dimension 1, suggesting the magnitude of asymmetric occupational mobility between non-manual and manual occupational categories increased relatively more than that of asymmetric occupational mobility within both non-manual and manual occupational categories. From 1965 to 1975, asymmetric occupational mobility along Dimension 1 increased, while asymmetric occupational mobility along Dimension 2 decreased. Thus, from 1965 to 1975 the magnitudes of asymmetric occupational mobility within both non-manual and manual occupational categories increased, while such mobility between non-manual and manual occupational categories decreased. From 1975 to 1985, asymmetric occupational mobility along Dimension 1 decreased but was almost constant, while asymmetric occupational mobility along Dimension 2 decreased, suggesting the magnitudes of asymmetric occupational mobility between non-manual and manual occupational categories decreased relatively more than that within both non-manual and manual occupational categories.

## 5. Discussion

An asymmetric MDS model and corresponding algorithm to analyze two- mode three-way proximities have been introduced. The model and the algorithm are extensions of Okada and Imaizumi's (1987) for one-mode two-way asymmetric proximities, and were successfully used to analyze intergenerational occupational mobility from father to son among eight occupational categories.

As shown in the asymmetry weight configuration of Figure 6, four points seem to be close to a diagonal line. The indeterminacy of the orientation of the dimensions will be further considered here. As noted earlier, the best $S$ value obtained in the two-dimensional space was 0.378. The dimensions of the resultant two-dimensional asymmetry weight configuration were rotated 10 degrees, clockwise and counterclockwise. Then the common object configuration **X**, radii and symmetry weights which minimize $S$ were derived for both of the two rotated asymmetry weight configurations. For the clockwise rotated case, $S$ value was 0.448, for the counterclockwise case, 0.458, thus suggesting that the indeterminacy of the orientation of the dimensions of the present solution is not serious.

The present asymmetric MDS model for two-mode three-way proximities is an extension of the predecessor (Okada and Imaizumi 1987). Alternative extensions were pointed out by a reviewer. For example, when $\mathbf{u}_{it} = u_t$, a simpler model is derived, in which an object has the same ellipse through all sources in the configuration of objects. In this model $v_{jki}$ and $v_{kji}$ in Equation (2.3) respectively become $v_{jk}$ and $v_{kj}$. When $u_{it} = u_i$, another, simpler model is possible, where an object has circles of different radii for each source in the configuration of objects. In this model, both $v_{jki}$ and $v_{kji}$ become $v_i$. Although capable of extensions, these simpler models cannot fully address the motivation for developing the present model, to model individual differences in asymmetric proximity relationships. These two simpler models can be combined into one where $u_{it}$ is multiplicatively decomposed into two terms, $y_i$ and $u_t$, i.e., $u_{it} = y_i u_t$. In this model all objects have an identical ratio of the lengths of semiaxes through all objects, so that relative magnitudes of asymmetric proximity relationships along all dimensions are same for all sources. In this model, $u_t$ is represented as a point in an asymmetry weight configuration, and $y_i u_t$ (where $u_t$, $t = 1, \ldots ,p$) are uniformly stretched or shrunk over $p$ dimensions by $y_i$ and are represented on a line passing through the origin. Although four points of the asymmetric weight configuration shown in Figure 6 are not exactly on a line passing through the origin, they nearly located on the line, suggesting this simpler model rather than the present one is sufficient for the data shown in Table 2. But if this model were used to analyze the data, the differences in magnitudes along the two dimensions over the four years would not be manifest.

A more general model employs $w_{it}$ instead of $w_i$ (Okada and Imaizumi 1992), and the common object configuration **X** is not uniformly stretched or shrunk over $p$ dimensions but is instead differentially stretched or shrunk along each of the $p$ dimensions, as in the INDSCAL model (Carroll and Chang 1970). This alternative model can deal with individual differences in symmetric proximity relationships according not only to magnitude but also to the magnitude of each dimension. Although this model seems interesting as an asymmetric MDS model for two-mode three-way proximities, the major intent of the present paper is, as noted earlier, to address individual differences in *asymmetric* rather than symmetric proximity relationships. One deficient of this more general model is the orientation of dimensions. In the present model the orientation of dimensions is determined by $u_{it}$, based the individual differences in asymmetric proximity relationships. In the more general model with $w_{it}$, the orientation of dimensions is determined not only by the individual differences in asymmetric proximity relationships but also by the individual differences in symmetric proximity relationships, unless $w_{it'}$ and $u_{it}$ are respectively applied to discrepantly oriented dimensions $t'$ and $t$. While Zielman's (1991) two-mode three-way model does not have this problem, the two-mode three-way extension of the slide vector model (Zielman and Heiser 1991, 1993) does seem to. A potentially promising model more general than the present one also seems possible and allows each source an idiosyncratic orientation of the dimensions for the configuration of objects. Each source has its own ellipses with axes having an idiosyncratic orientation in the object configuration. This model does not have the problem of determining the orientation of dimensions.

Although the asymmetry weights $u_{it}$ were assumed nonnegative, another more general model which allows negative $u_{it}$ can also be introduced.

Moreover, the present algorithm can be applied to this more general model with only minor modifications. With $u_{it}$ allowed to be negative, we simply alter Equation (2.4) to

$$v_{jki} = \text{sign}\,(z_{jki})\ |\ z_{jki}\ |^{\frac{1}{2}}\ , \tag{5.1}$$

and Equation (2.5) to

$$z_{jki} = \frac{w_i^2}{\sum_{s=1}^{p} (\frac{x_{js} - x_{ks}}{u_{is}})^2} \sum_{t=1}^{p} \text{sign}\,(u_{it})(x_{kt} - x_{jt})^2\ . \tag{5.2}$$

When $u_{it}$ is assumed nonnegative, it is implicitly assumed that the direction of asymmetry is same for all $N$ sources, i.e., when $s_{jki} > s_{kji}$, it is automatically assumed that $s_{jkh} > s_{kjh}$. The more general model can depict the situation where the direction of asymmetry differs among $N$ sources (Zielman 1991, p. 12) and can also expand the horizon of the application of the present asymmetric MDS.

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