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Three-mode models for multivariate longitudinal data

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Multivariate longitudinal data are characterized by three modes: variables, occasions and subjects. Three-mode models are described as special cases of a linear latent variable model. The assumption of measurement invariance across occasions yields three-mode models that are suited for the analysis of multivariate longitudinal data. These so-called longitudinal three-mode models include autoregressive models and latent curve models as special cases. Empirical data from the field of industrial psychology are used in an example of how to test substantive hypotheses with the longitudinal, autoregressive and latent curve three-mode models.

1. Introduction

Multivariate longitudinal data consist of participants' (subjects') scores on multiple variables (tests or test items) on different occasions. Here we only consider the case where scores on the same variables are obtained on many occasions. Three-mode models are models for sets of data that are characterized by three modes. Multivariate longitudinal data are of this form, the modes being variables, occasions and subjects.

Three-mode models originate from Tucker's (1966) three-mode principal components analysis. In principal components analysis all modes are considered fixed. Considering the subjects mode random, Bloxom (1968) developed three-mode factor analysis. Bentler, Poon and Lee (1988) extended this work to what may be called three-mode covariance structure analysis, or three-mode structural equation modelling. Oort (1999) proposed investigating the three-mode structure of all parameters that feature in structural equation models, including the parameters of the mean structures. He also mentioned a special class of three-mode models, geared to multivariate longitudinal data. The purpose of the present paper is to examine these so-called longitudinal three-mode models further.

First, we describe the general mean and covariance structures for multivariate longitudinal data (after Tisak & Meredith, 1990, who gave the covariance structure only). These mean and covariance structures fit into Oort's (1999) framework of stochastic three-mode models (S3MMs). We then discuss the requirement of measurement invariance. Imposition of measurement invariance yields the longitudinal three-mode models (L3MMs) mentioned

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above. So L3MMs are special cases of S3MMs, which themselves are special cases of the linear latent variable model commonly used in structural equation modelling. Further special cases of L3MMs are autoregressive three-mode models (AR3MMs) and latent curve three-mode models (LC3MMs).

AR3MMs result from imposing autoregressive structures on the mean and covariance structures of the L3MM. Autoregressive models have a long history that goes back to descriptions of simplex models by Guttman (1954) and Jöreskog (1970). Yet these models are for univariate longitudinal data, that is, the case of the repeated measurement of just a single variable. Jöreskog (1979) and Swaminathan (1984) describe autoregressive models for the repeated measurement of multiple variables, though they model variances and covariances only, and not the means. Mandys, Dolan and Molenaar (1994) give an autoregressive model for the mean structure in addition to the covariance structure, but restrict themselves to the univariate case. Here we describe AR3MMs for the mean and covariance structures of multivariate longitudinal data.

LC3MMs result from imposing a latent curves structure on the L3MM. Meredith and Tisak (1990) are often mentioned as the originators of latent curve analysis. Latent (growth) curve modelling has now become very popular, as can be judged from the textbook of Duncan, Duncan, Strycker, Li and Alpert (1999). However, in most applications of latent curve models only polynomial functions are considered to describe the development of subjects' scores. The LC3MMs described below are developed by extending Browne's (1993) structured latent curve models for the repeated measurement of a single observed variable to the case of multiple latent variables. Imposition of such latent curve models on the mean and covariance structure of the common factors of the L3MM produces LC3MMs.

In practice, multivariate longitudinal data are gathered to answer specific research questions. These questions often concern the development of some trait or ability, rather than the structure of data. So, after deciding on an appropriate longitudinal structure, researchers may want to test, for example, whether the factor means are equal across occasions, or whether the factor means can be described by some trend. Therefore, we give some examples of how to test such hypotheses within the framework of L3MMs, AR3MMs and LC3MMs. All models discussed are illustrated by fitting them to some empirical data from the field of industrial psychology.

2. Mean and covariance structures for multivariate longitudinal data

Suppose R latent traits are measured with K observed variables on J occasions. For a single occasion j, the observed scores are modelled by

$$\mathbf{x}_j = \boldsymbol{\tau}_j + \boldsymbol{\Lambda}_j \boldsymbol{\xi}_j + \boldsymbol{\Delta}_j \boldsymbol{\varepsilon}_j, \tag{1}$$

where $\mathbf{x}_j$ is a random K-vector of observed scores for an arbitrary subject on occasion j, $\boldsymbol{\xi}_j$ is a random R-vector of scores on the latent traits or common factors, $\boldsymbol{\varepsilon}_j$ is a random K-vector of scores on the residual factors, $\boldsymbol{\tau}_j$ is a K-vector of intercepts, $\boldsymbol{\Lambda}_j$ is a $K \times R$ matrix of (common) factor loadings, and $\boldsymbol{\Delta}_j$ is a $K \times K$ diagonal matrix of (residual) factor loadings. (Hereinafter, the terms 'common factor loadings' ($\boldsymbol{\Lambda}_j$) and 'residual factor loadings' ($\boldsymbol{\Delta}_j$) are used as shorthand to refer to the regression coefficients of the observed variables on the common factors and residual factors, respectively.) The residual factors are not correlated with each other, are not correlated with the common factors, and have zero means. It follows that the means and covariances of the observed variables, on occasion j, are given by

$$E(\mathbf{x}_j) = \boldsymbol{\mu}_j = \boldsymbol{\tau}_j + \boldsymbol{\Lambda}_j \boldsymbol{\kappa}_j \tag{2}$$

and

$$\mathrm{Cov}(\mathbf{x}_j, \mathbf{x}_j') = \boldsymbol{\Sigma}_{jj} = \boldsymbol{\Lambda}_j \boldsymbol{\Phi}_{jj} \boldsymbol{\Lambda}_j' + \boldsymbol{\Delta}_j \boldsymbol{\Theta}_{jj} \boldsymbol{\Delta}_j', \tag{3}$$

where $\boldsymbol{\kappa}_j$ is an R-vector of common factor means, $\boldsymbol{\Phi}_{jj}$ is an $R \times R$ symmetric matrix containing the variances and covariances of the common factors, and $\boldsymbol{\Theta}_{jj}$ is a $K \times K$ diagonal matrix containing the variances of the residual factors.

For J occasions, the single occasion matrices are collected in larger matrices:

$$\mathbf{x} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\xi} + \boldsymbol{\Delta}\boldsymbol{\varepsilon}, \tag{4}$$

where $\mathbf{x}$, $\boldsymbol{\tau}$, $\boldsymbol{\Lambda}$, $\boldsymbol{\xi}$, $\boldsymbol{\Delta}$ and $\boldsymbol{\varepsilon}$ are partitioned vectors and matrices. Vectors $\mathbf{x}$, $\boldsymbol{\tau}$, $\boldsymbol{\xi}$ and $\boldsymbol{\varepsilon}$ consist of stacked $\mathbf{x}_j$, $\boldsymbol{\tau}_j$, $\boldsymbol{\xi}_j$ and $\boldsymbol{\varepsilon}_j$ vectors. Vectors $\mathbf{x}$, $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$ are $JK \times 1$, and vector $\boldsymbol{\xi}$ is $JR \times 1$. Matrix $\boldsymbol{\Lambda}$ is a $JK \times JR$ block diagonal matrix,

$$\boldsymbol{\Lambda} = \begin{vmatrix} \boldsymbol{\Lambda}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Lambda}_J \end{vmatrix}, \tag{5}$$

and matrix $\boldsymbol{\Delta}$, having the same structure but consisting of diagonal $\boldsymbol{\Delta}_j$ matrices, is a $JK \times JK$ diagonal matrix. Means and covariances of the K scores on J occasions are given by

$$E(\mathbf{x}) = \boldsymbol{\mu} = \boldsymbol{\tau} + \boldsymbol{\Lambda}\boldsymbol{\kappa} \tag{6}$$

and

$$\mathrm{Cov}(\mathbf{x}, \mathbf{x}') = \boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}' + \boldsymbol{\Delta}\boldsymbol{\Theta}\boldsymbol{\Delta}', \tag{7}$$

where $\boldsymbol{\kappa}$ is a JR-vector consisting of stacked $\boldsymbol{\kappa}_j$ vectors, $\boldsymbol{\Phi}$ is a $JR \times JR$ symmetric matrix consisting of $\boldsymbol{\Phi}_{jj'}$ matrices, and $\boldsymbol{\Theta}$ is a $JK \times JK$ symmetric matrix consisting of $\boldsymbol{\Theta}_{jj}$ matrices ($\boldsymbol{\Phi}_{jj'} = \boldsymbol{\Phi}_{j'j}$; $\boldsymbol{\Theta}_{jj'} = \boldsymbol{\Theta}_{j'j}$; $j, j' = 1, \ldots, J$), That is,

$$\boldsymbol{\Phi} = \begin{vmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \cdots & \boldsymbol{\Phi}_{1J} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} & \cdots & \boldsymbol{\Phi}_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Phi}_{J1} & \boldsymbol{\Phi}_{J2} & \cdots & \boldsymbol{\Phi}_{JJ} \end{vmatrix}, \tag{8}$$

where a $\boldsymbol{\Phi}_{jj'}$ matrix contains the covariances of the common factors on occasion j with the common factors on occasion j'. Residual factors of different observed variables are uncorrelated, but residual factors of the same observed variables on different occasions are allowed to correlate. Therefore all $\boldsymbol{\Theta}_{jj'}$ matrices are diagonal.

2.1. Measurement invariance

As it is our goal to test substantive hypotheses about the common factors, we have to make sure that the content (meaning) of these factors is the same across occasions. We therefore require that all measurement parameters are invariant across occasions. Measurement parameters are the factor loadings in the $\mathbf{\Lambda}$ matrix, the intercepts in the $\boldsymbol{\tau}$ vector, and the residual factor loadings in the $\mathbf{\Delta}$ matrix. The factor loadings represent the degree to which the $\mathbf{x}_j$ variables are indicative of the $\boldsymbol{\xi}_j$ factors, and thus concern the content of the factors. The intercepts in vector $\boldsymbol{\tau}$ relate to the level of factors. If the observed variables are test items, then the factor loadings and intercepts can be interpreted as item discrimination and (the reciprocal of) item difficulty, respectively (Oort, 1996).

Common factor means and variances can only be compared across occasions if common factor loadings and intercepts are invariant across occasions. That is, $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2 = \cdots = \mathbf{\Lambda}_J$, and $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \cdots = \boldsymbol{\tau}_J$, or

$$\mathbf{\Lambda} = (\mathbf{I}_{J \times J} \otimes \mathbf{\Lambda}_0) \tag{9}$$

and

$$\boldsymbol{\tau} = (\mathbf{u} \otimes \boldsymbol{\tau}_0), \tag{10}$$

where $\mathbf{\Lambda}_0$ is a $K \times R$ matrix of factor loadings, $\boldsymbol{\tau}_0$ is a $K \times 1$ vector of intercepts, $\mathbf{I}_{J \times J}$ is a $J \times J$ identity matrix, $\mathbf{u}$ is a $J \times 1$ vector of ones, and the $\otimes$ symbol denotes the Kronecker product (or direct product). The remaining measurement parameters, the residual factor loadings in the $\mathbf{\Delta}$ matrix, do not concern the common factors. The invariance of these parameters is therefore not required for testing hypotheses about common factors. However, in most cases the across-occasion invariance of the $\mathbf{\Delta}_j$ matrices,

$$\mathbf{\Delta} = (\mathbf{I}_{J \times J} \otimes \mathbf{\Delta}_0), \tag{11}$$

is satisfied as a matter of the model identification requirements to be discussed below.

The measurement invariance restrictions of equations (9) and (10) are a prerequisite for the analysis of the mean and covariance structures of repeatedly measured common factors. Therefore, the most general model for the analysis of multivariate longitudinal data to be presented here is the linear latent variable model (LLVM) of equation (4), with mean and covariance structures given by (6) and (7), and with the measurement invariance restrictions of (9) and (10). We call this model the longitudinal three-mode model (L3MM), as it is a special case of the general three-mode models described by Oort (1999).

2.2. Three-mode models

The general three-mode models discussed by Oort (1999) are called stochastic three-mode models (S3MMs), to contrast them with the (non-stochastic) three-mode principal components models that originate from the work of Tucker (1966; see Kroonenberg, 1983; Kiers, 1991). The S3MMs are an extension of the three-mode factor analysis models of Bloxom (1968) and Bentler and Lee (1978, 1979), and the composite direct product models of Browne (1984).

S3MMs can be applied to three-mode data if one of the modes can be considered random. One example is the analysis of multitrait–multimethod data where the modes are traits, methods and subjects, where the subjects are generally taken to be random. With multivariate longitudinal data the modes are variables, occasions and subjects.

The S3MMs are models for mean and covariance structures. They can be described by restrictions on the parameter matrices that feature in the mean and covariance structures of the LLVM, given by (6) and (7). The LLVM becomes a three-mode model after imposing at least one of the following constraints:

$$\mathbf{\Lambda} = (\mathbf{\Lambda}_{\mathrm{S}} \otimes \mathbf{\Lambda}_{\mathrm{F}})\mathbf{\Gamma}, \tag{12}$$

$$\boldsymbol{\tau} = (\boldsymbol{\tau}_{\mathrm{S}} \otimes \boldsymbol{\tau}_{\mathrm{F}}), \tag{13}$$

$$\mathbf{\Delta} = (\mathbf{\Delta}_{\mathrm{S}} \otimes \mathbf{\Delta}_{\mathrm{F}}), \tag{14}$$

$$\mathbf{\Theta} = (\mathbf{\Theta}_{\mathrm{S}} \otimes \mathbf{\Theta}_{\mathrm{F}}), \tag{15}$$

where $\mathbf{\Lambda}_{\mathrm{S}}$ is $J \times Q$, $\mathbf{\Lambda}_{\mathrm{F}}$ is $K \times R$, $\mathbf{\Gamma}$ is $QR \times P$, $\boldsymbol{\tau}_{\mathrm{S}}$ is $J \times 1$, $\boldsymbol{\tau}_{\mathrm{F}}$ is $K \times 1$, $\mathbf{\Delta}_{\mathrm{S}}$ is $J \times J$, $\mathbf{\Delta}_{\mathrm{F}}$ is $K \times K$, $\mathbf{\Theta}_{\mathrm{S}}$ is $J \times J$, and $\mathbf{\Theta}_{\mathrm{F}}$ is $K \times K$. If $\mathbf{\Gamma}$ is diagonal, then $P = QR$, and the following constraints may be imposed as well:

$$\mathbf{\Phi} = (\mathbf{\Phi}_{\mathrm{S}} \otimes \mathbf{\Phi}_{\mathrm{F}}), \tag{16}$$

$$\boldsymbol{\kappa} = (\boldsymbol{\kappa}_{\mathrm{S}} \otimes \boldsymbol{\kappa}_{\mathrm{F}}), \tag{17}$$

where $\mathbf{\Phi}_{\mathrm{S}}$ is $Q \times Q$, $\mathbf{\Phi}_{\mathrm{F}}$ is $R \times R$, $\boldsymbol{\kappa}_{\mathrm{S}}$ is $Q \times 1$, and $\boldsymbol{\kappa}_{\mathrm{F}}$ is $R \times 1$.

The subscripts S and F stand for slow mode and fast mode, referring to how the data are organized. The fast mode is the mode whose levels change fastest in the data set, and the slow mode is the mode whose levels change slowest. With multivariate longitudinal data sets, the scores of the subjects are usually given first for all variables on the first occasion, then for all variables on the second occasion, and so on. Such an organization would associate the slow mode with the occasions, and the fast mode with the variables. Alternative terms for slow and fast modes are outer and inner modes.

For the meaning of the restrictions of (12)–(17), and the interpretation of the coefficients of the matrices and vectors that feature in these restrictions, the reader is referred to Oort (1999). Here we only point out that the L3MM is a special case of the S3MM, since the measurement invariance constraints of (9) and (10) are special cases of the Kronecker-product constraints of (12) and (13) (that is, $\mathbf{\Lambda}_{\mathrm{S}} = \mathbf{I}$, $\mathbf{\Lambda}_{\mathrm{F}} = \mathbf{\Lambda}_0$, $\mathbf{\Gamma} = \mathbf{I}_{JK \times JK}$, $\boldsymbol{\tau}_{\mathrm{S}} = \mathbf{u}$, $\boldsymbol{\tau}_{\mathrm{F}} = \boldsymbol{\tau}_0$, $\mathbf{\Delta}_{\mathrm{S}} = \mathbf{I}$ and $\mathbf{\Delta}_{\mathrm{F}} = \mathbf{\Delta}_0$). This is useful for the discussion of the identification and estimation of the L3MM parameters.

2.3. *Identification and estimation*

As the L3MM and S3MM are special cases of the LLVM, the general guidelines for the identification of structural equation models apply (Bollen, 1989). These guidelines yield identification conditions that are either necessary or sufficient. To evaluate the identification of a particular model the procedure of Bekker, Merckens & Wansbeek (1994) can be used: if and only if the kernel of the Jacobian matrix for a particular model is empty, then that model is identified. This rule is both necessary and sufficient for global identification. It can be evaluated symbolically with computer programs for symbolic computation like Maple (Char *et al.*, 1991) and Mathematica (Wolfram, 1991)—see Bekker *et al.* (1994) or the book review by Rigdon (1997) for a summary of the identification procedure.

One necessary identification condition is that all latent variables have a scale and an origin. In the L3MM, latent variables are the common factors $\boldsymbol{\xi}$ and the residual factors $\boldsymbol{\varepsilon}$. Scales and origins can be imposed through the measurement parameters (in $\mathbf{\Lambda}$, $\boldsymbol{\tau}$ and $\mathbf{\Delta}$), or through the other structural parameters (in $\mathbf{\Phi}$, $\boldsymbol{\kappa}$ and $\mathbf{\Theta}$). In the L3MM the $\mathbf{\Lambda}$ and $\boldsymbol{\tau}$ matrices are restricted as in (9) and (10). Consequently, scales for the ξ factors can be provided either by fixing one factor loading per factor at a non-zero value (that is, one element in each column of $\mathbf{\Lambda}_0$), or by fixing the R factor variances on one occasion at a non-zero value (the diagonal elements of one of the $\mathbf{\Phi}_{jj}$ matrices). Origins for the $\boldsymbol{\xi}$ factors can be provided either by fixing R intercepts, one intercept per factor (in $\boldsymbol{\tau}_0$), or by fixing the R factor means on one occasion (in $\boldsymbol{\kappa}_j$). Scales for the $\boldsymbol{\varepsilon}$ factors can be provided either by fixing the residual factor loadings (the $\mathbf{\Delta}$ diagonal) or by fixing the residual factor variances (the $\mathbf{\Theta}$ diagonal) at a non-zero value. As there are as many residual factors as observed variables, the $\mathbf{\Delta}$ coefficients cannot be interpreted relative to each other. In practice the $\mathbf{\Delta}$ matrix is mostly set to identity, satisfying the invariance restriction of (11) as a matter of course. See Oort (1999) for a more detailed discussion of the identification of S3MMs.

Assuming a multivariate normal distribution for the observed variables $\mathbf{x}$, and assuming that the model is identified, maximum likelihood (ML) estimates of all parameters can be obtained by minimizing the following fit function:

$$F(\mathbf{m}, \mathbf{S}, \boldsymbol{\mu}, \mathbf{\Sigma}) = (\mathbf{m} - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{m} - \boldsymbol{\mu}) + \log|\mathbf{\Sigma}| - \log|\mathbf{S}| + \text{trace}\,(\mathbf{S}\mathbf{\Sigma}^{-1}) - JK, \tag{18}$$

where $\mathbf{m}$ is the vector of observed sample means of $\mathbf{x}$, and $\mathbf{S}$ is the matrix of observed sample variances and covariances of $\mathbf{x}$ (see Bollen, 1989; or Jöreskog & Sörbom, 1996). Minimization of the ML function gives parameter estimates that are unbiased, scale-invariant, and scale-free. Note that the latter properties do not generally hold for models that include (non)linear equality or inequality constraints, such as some of the models discussed later. The ML estimation method also provides estimates of the asymptotic standard errors of the model parameters, and a chi-square test of overall goodness of fit of the model.

There are weaker assumptions than multivariate normality of $\mathbf{x}$ that still justify parameter estimation through minimization of the ML function (Browne & Shapiro, 1988). Alternatively, parameter estimates can be obtained through other procedures, but not all procedures share all of the useful properties of ML estimation (see Bollen, 1989; Browne & Arminger, 1995).

The basic L3MM can be fitted with standard software for structural equation modelling such as LISREL (Jöreskog & Sörbom, 1996) and EQS (Bentler, 1995). However, neither of these computer programs is as versatile as Mx (Neale, Boker, Xie & Maes, 1999), which is freely available through the Internet (http://views.vcu.edu/mx). Some special cases of the L3MM that will be discussed below involve complicated constraints and cannot be fitted with LISREL or EQS.

3. Longitudinal three-mode models

As explained above, substitution of the measurement invariance equations (9) and (10) into the mean and covariance structures of (6) and (7) yields the mean and covariance structures of the L3MM:

$$E(\mathbf{x}) = \boldsymbol{\mu} = (\mathbf{u} \otimes \boldsymbol{\tau}_0) + (\mathbf{I} \otimes \mathbf{\Lambda}_0)\boldsymbol{\kappa} \tag{19}$$

and

$$\text{Cov}(\mathbf{x}, \mathbf{x}') = \mathbf{\Sigma} = (\mathbf{I} \otimes \mathbf{\Lambda}_0)\,\mathbf{\Phi}\,(\mathbf{I} \otimes \mathbf{\Lambda}_0') + \mathbf{\Delta\Theta\Delta}. \tag{20}$$

Sometimes it is convenient to write the $\mathbf{\Phi}$ matrix as

$$\mathbf{\Phi} = \mathbf{\Gamma}\mathbf{\Phi}^*\mathbf{\Gamma}, \tag{21}$$

where $\text{diag}(\mathbf{\Phi}^*) = \mathbf{I}_{JR \times JR}$, and $\mathbf{\Gamma}$ is diagonal and free. The $\mathbf{\Phi}^*$ coefficients can then be interpreted as correlation coefficients.

3.1. Hypothesis testing with the L3MM

Hypotheses about the common factors can be tested by further restricting (19) and (20), and comparing the fit of the restricted models with the fit of the unrestricted model through the likelihood-ratio or chi-square difference test (Bollen, 1989). For example, the hypothesis of equal variances across occasions can be investigated by writing the $\mathbf{\Phi}$ matrix as in (21), imposing

$$\mathbf{\Gamma} = \mathbf{I}_{J \times J} \otimes \mathbf{\Gamma}_0 \tag{22}$$

and allowing $\mathbf{\Phi}^*$ to be free. $\mathbf{\Gamma}_0$ is then a diagonal $R \times R$ matrix containing the invariant standard deviations of the common factors.

The hypothesis of equal correlations across occasions can be investigated by imposing

$$\mathbf{\Phi}^* = \mathbf{\Phi}^*_{\mathrm{S}} \otimes \mathbf{\Phi}^*_{\mathrm{F}} \tag{23}$$

instead, allow the diagonal matrix $\mathbf{\Gamma}$ to be free, and imposing a banded structure on $\mathbf{\Phi}^*_{\mathrm{S}}$. A banded structure is a structure of a symmetric matrix where all elements on the same diagonal have the same value. That is, covariances (or correlations) of the same lag are equal to each other: all $\varphi_{\mathrm{S}jj'}$ with equal $|j - j'|$ are equal $(j, j' = 1, \ldots, J)$.

If neither the equal variances hypothesis of (21) nor the equal correlations hypothesis of (22) is rejected, the hypothesis of equal covariances can be investigated by imposing

$$\mathbf{\Phi} = \mathbf{\Phi}_{\mathrm{S}} \otimes \mathbf{\Phi}_{\mathrm{F}}, \tag{24}$$

with $\mathbf{\Phi}_{\mathrm{S}}$ banded. Unless one of the $\mathbf{\Phi}_{\mathrm{S}}$ and $\mathbf{\Phi}_{\mathrm{F}}$ matrices has already been restricted to provide scales for the common factors, the Kronecker-product restriction introduces a new indeterminacy that must be removed by fixing one of the elements of either of the constituent matrices $\mathbf{\Phi}_{\mathrm{S}}$ and $\mathbf{\Phi}_{\mathrm{F}}$ at a non-zero value.

Substantive hypotheses generally concern the means of the common factor, rather than variances, covariances or correlations. The hypothesis of equal factor means is investigated by imposing

$$\boldsymbol{\kappa} = \mathbf{u} \otimes \boldsymbol{\kappa}_{\mathrm{F}}. \tag{25}$$

Another hypothesis that may be interesting in practical applications of the L3MM is whether the common factor means can be described by a linear trend:

$$\boldsymbol{\kappa} = \mathbf{u} \otimes \mathbf{a} + \mathbf{t} \otimes \mathbf{b}, \tag{26}$$

where $\mathbf{a}$ is an R-vector of intercepts, $\mathbf{b}$ is an R-vector of slope parameters, and $\mathbf{t}$ is a J-vector with some coding for the time of the occasion, for example, $\mathbf{t}' = |1\,2 \ldots J|$. With equation (26), the equality of factor means across occasions can be investigated by fixing the slopes at zero ($\mathbf{b} = \mathbf{0}$), which then reduces (26) to (25).

The hypotheses of (22)–(25) all are special cases of the Kronecker-product constraints of the S3MM ((12)–(17)). The linear mean trend hypothesis of (26) can also be written as a special case of the Kronecker-product constraint of (17). If origins for the common factors are provided by fixing the factor means of the first occasion at zero ($\boldsymbol{\kappa}_1 = \mathbf{0}$, $\boldsymbol{\tau}_0$ free), and the time of the occasion is coded $t_j = j - 1$, then (26) simplifies to (17) with $\boldsymbol{\kappa}_{\mathrm{S}} = \mathbf{t}$ and $\boldsymbol{\kappa}_{\mathrm{F}} = \mathbf{b}$.

Of course it is also possible to test hypotheses about the residual variances, covariances or correlations. If we write

$$\mathbf{\Theta} = \mathbf{\Delta}\mathbf{\Theta}^{*}\mathbf{\Delta}, \tag{27}$$

where $\operatorname{diag}(\mathbf{\Phi}^{*}) = \mathbf{I}$, then the hypothesis of equal residual variances across occasions is investigated by imposing the constraint of equation (11). The hypothesis of equal residual correlations can be investigated by imposing

$$\mathbf{\Theta}^{*} = \mathbf{\Theta}_{\mathrm{S}}^{*} \otimes \mathbf{\Theta}_{\mathrm{F}}^{*}, \tag{28}$$

allow the diagonal matrix $\mathbf{\Delta}$ to be free, and imposing a banded structure on $\mathbf{\Theta}_{\mathrm{S}}^{*}$. The hypothesis of equal residual covariances is investigated by imposing

$$\mathbf{\Theta} = \mathbf{\Theta}_{\mathrm{S}} \otimes \mathbf{\Theta}_{\mathrm{F}}, \tag{29}$$

where $\mathbf{\Theta}_{\mathrm{S}}$ is banded and $\mathbf{\Theta}_{\mathrm{F}}$ is diagonal.

In the L3MM the longitudinal nature of the data is only taken into account to the extent of assuming measurement invariance. The hypotheses that have subsequently been discussed only concern the equality of factor means, variances and covariances. Later we test hypotheses of some well-known longitudinal structures. The hypothesis that the common factors conform to an autoregressive model is tested by imposing an autoregressive structure on the $\mathbf{\Phi}$ and $\boldsymbol{\kappa}$ matrices of the L3MM, thus yielding an autoregressive three-mode model (AR3MM). Likewise, the hypothesis that the common factors conform to a latent (growth) curve model is tested by imposing a latent curve structure on the $\mathbf{\Phi}$ and $\boldsymbol{\kappa}$ matrices of the L3MM, yielding a latent curve three-mode model (LC3MM).

4. Autoregressive three-mode models

An AR3MM is an L3MM with restrictions on the $\boldsymbol{\kappa}$ and $\mathbf{\Phi}$ matrices that conform to an autoregressive structure for means and covariances of the common factors (equations (35) and (36) below).

In describing the autoregressive model for the common factors, we build on the work of Jöreskog (1979), Dwyer (1983, Chapter 11), Swaminathan (1984), and Mandys *et al.* (1994). Mandys *et al.* only model the repeated measurement of a single observed variable, and the other authors do not model mean structures. However, the accumulated work of these authors is easily extended to models for mean and covariance structures of multiple latent variables.

4.1. Autoregressive model for the common factors

In the basic AR3MM, a first-order autoregressive model is assumed for the common factors. On occasion j, an arbitrary individual's scores on the R common factors are given by:

$$\boldsymbol{\xi}_1 = \boldsymbol{\zeta}_1 \qquad \text{and} \qquad \boldsymbol{\xi}_j = \mathbf{B}_{j,j-1}\boldsymbol{\xi}_{j-1} + \boldsymbol{\zeta}_j, \quad j = 2, \ldots, J, \tag{30}$$

where $\mathbf{B}_{j,j-1}$ is an $R \times R$ diagonal matrix of regression coefficients, and $\boldsymbol{\zeta}_j$ is a random R-vector of scores on the innovation factors. Innovation factors $\boldsymbol{\zeta}_j$ represent everything that happened between occasions j and $j-1$, uncorrelated with $\boldsymbol{\xi}_{j-1}$. At the first occasion the innovation factor scores and the common factor scores coincide. One could say that the first occasion innovation factors $\boldsymbol{\zeta}_1$ represent everything that happened before the first occasion.

Assuming that the innovation factors are correlated neither with the common factors nor with each other, the within-occasion means and covariances of the common factors are

$$E(\boldsymbol{\xi}_1) = \boldsymbol{\kappa}_1 = \boldsymbol{\alpha}_1, \qquad E(\boldsymbol{\xi}_j) = \boldsymbol{\kappa}_j = \boldsymbol{\alpha}_j + \mathbf{B}_{j,j-1}\boldsymbol{\kappa}_{j-1}, \tag{31}$$

$$\mathrm{Cov}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_1') = \boldsymbol{\Phi}_{11} = \boldsymbol{\Psi}_{11}, \qquad \mathrm{Cov}(\boldsymbol{\xi}_j, \boldsymbol{\xi}_j') = \boldsymbol{\Phi}_{jj} = (\mathbf{I} - \mathbf{B}_{j,j-1})^{-1}\boldsymbol{\Psi}_{jj}(\mathbf{I} - \mathbf{B}_{j,j-1}')^{-1}, \tag{32}$$

for $j = 2, \ldots, J$. $\boldsymbol{\alpha}_j$ is an R-vector containing the means of the innovation factors, and $\boldsymbol{\Psi}_{jj}$ is a diagonal $R \times R$ matrix containing the variances of the innovation factors within occasion j (symmetric $\boldsymbol{\Psi}_{jj}$ matrices will be discussed below). On the first occasion, the factors are allowed to correlate, so that $\boldsymbol{\Psi}_{11}$ is a symmetric matrix of variances and covariances of the exogenous factors.

Equation (30) gives the model for the common factor scores on occasion j. For J occasions, the single occasion matrices are collected in partitioned matrices:

$$\boldsymbol{\xi} = \mathbf{B}\boldsymbol{\xi} + \boldsymbol{\zeta} \Leftrightarrow \boldsymbol{\xi} = (\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\zeta}), \tag{33}$$

where $\boldsymbol{\xi}$ is defined as before, $\mathbf{I}$ is a $JR \times JR$ identity matrix, $\boldsymbol{\zeta}$ is a partitioned vector consisting of stacked $\boldsymbol{\zeta}_j$ vectors, and $\mathbf{B}$ is a square $JR \times JR$ matrix,

$$\mathbf{B} = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ \mathbf{B}_{21} & 0 & & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & \mathbf{B}_{J,J-1} & 0 \end{vmatrix}. \tag{34}$$

Note that, in this basic autoregressive model, the $\mathbf{B}$ matrix contains $\mathbf{B}_{j,j-1}$ matrices only, and that all $\mathbf{B}_{j,j-1}$ matrices are diagonal. Other $\mathbf{B}$ structures will be discussed below.

Across occasions, the innovation factors are correlated with neither common factors nor innovation factors. The means and covariances of the common factors thus are

$$E(\boldsymbol{\xi}) = \boldsymbol{\kappa} = (\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\alpha}, \tag{35}$$

$$\mathrm{Cov}(\boldsymbol{\xi}, \boldsymbol{\xi}') = \boldsymbol{\Phi} = (\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\Psi}(\mathbf{I} - \mathbf{B}')^{-1}, \tag{36}$$

where $\boldsymbol{\Psi}$ is a block diagonal $JR \times JR$ matrix,

$$\boldsymbol{\Psi} = \begin{vmatrix} \boldsymbol{\Psi}_{11} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\Psi}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\Psi}_{JJ} \end{vmatrix}, \tag{37}$$

with, as explained above, $\boldsymbol{\Psi}_{11}$ symmetric, and the other $\boldsymbol{\Psi}_{jj}$ diagonal ($j = 2, \ldots, J$). Other forms of the $\boldsymbol{\Psi}$ matrix will be discussed below.

4.2. AR3MM variations

The autoregressive model with the **B** and **Ψ** matrices as presented above is viewed as the basic autoregressive model. It can be extended to include synchronous correlations, synchronous effects, cross-lagged effects, and higher-order effects. The basic autoregressive model only features first-order autoregressive effects, represented in the **B** matrix (equation (34)) by the diagonal elements of the $\mathbf{B}_{j,j-1}$ matrices ($j = 2, \ldots, J$). These effects are interpreted as 'causal' or 'explanatory' effects of one common factor on the same common factor at the next occasion.

Covariances (or correlations) between different common factors on the first occasion are given by the coefficients in $\mathbf{\Psi}_{11}$ (equation (32)). Through the autoregression coefficients, these first occasion correlations also explain the correlations between the different common factors at later occasions. However, if the first occasion correlations do not provide sufficient explanation of the correlations at later occasions, then this may be due to correlations between different innovation factors. These correlations are called synchronous correlations. Synchronous correlations are within-occasion correlations between different innovation factors. Synchronous correlations are represented in the partitioned matrix **Ψ** (equation (37)) by the off-diagonal elements of the symmetric $\mathbf{\Psi}_{jj}$ matrices ($j = 2, \ldots, J$).

Instead of assuming that the common factors are merely correlated within occasions, one may assume causal relations among particular common factors. Causal effects within occasions are called synchronous effects, represented in the partitioned matrix **B** (equation (34)) by the off-diagonal elements of the $\mathbf{B}_{jj}$ matrices ($j = 1, \ldots, J$). Of course, if synchronous effects (in the $\mathbf{B}_{jj}$ matrices) are modelled, then corresponding synchronous correlations (in $\mathbf{\Psi}_{jj}$) have to be fixed at zero.

If we assume that one common factor is the cause of another common factor, then it is reasonable to assume that it takes some time for common factor r to affect common factor r'. Therefore, it is perhaps more appropriate to model lagged effects instead of synchronous effects. To distinguish between these lagged effects and autoregressive effects, which can be characterized as lagged effects between the same common factors, lagged effects between different factors are often called cross-lagged effects. Cross-lagged effects are causal effects between different common factors, across occasions. Cross-lagged effects are represented in the **B** (34) by the off-diagonal elements of $\mathbf{B}_{j,j-1}$ ($j = 2, \ldots, J$).

With first-order autoregressive effects and first-order cross-lagged effects, we assume that only common factor scores of the previous occasion $j - 1$ have a direct effect on common factor scores of occasion j. Yet it may be that common factor scores from earlier occasions also have direct effects. Second-order autoregressive effects are represented in **B** (34) by the diagonal elements of $\mathbf{B}_{j,j-2}$, and second-order cross-lagged effects are represented by the off-diagonal elements of $\mathbf{B}_{j,j-2}$ ($j = 3, \ldots, J$). Autoregressive and cross-lagged effects of order n are represented by the elements of $\mathbf{B}_{j,j-n}$ ($j = n + 1, \ldots, J$).

Of course, many of the effects discussed here can be combined to produce more AR3MM variations, although not all combinations make sense. Moreover, some of the extensions are incompatible, such as synchronous effects and synchronous correlations between the same pairs of factors.

4.3. Hypothesis testing with AR3MMs

Within the framework of the AR3MM, hypotheses can be tested by further restricting (35) and (36), and carrying out chi-square difference tests. For example, the hypothesis of equal regression coefficients across occasions can be investigated by restricting **B** by setting

$$\mathbf{B}_{21} = \cdots = \mathbf{B}_{J,J-1}, \tag{38}$$

the hypothesis that the variances (and possibly covariances) of the innovation factors are equal across occasions can be investigated by restricting $\boldsymbol{\Psi}$ by setting

$$\boldsymbol{\Psi}_{22} = \cdots = \boldsymbol{\Psi}_{J,J}, \tag{39}$$

and the hypothesis that the average amount of innovation is equal across occasions can be investigated by restricting $\boldsymbol{\alpha}$ by setting

$$\boldsymbol{\alpha}_2 = \cdots = \boldsymbol{\alpha}_J. \tag{40}$$

Hypotheses about the innovation factors are perhaps less interesting than hypotheses about the common factors. In the AR3MM the testing of such hypotheses is less straightforward than in the L3MM, as the $\boldsymbol{\kappa}$ vector of common factor means and the $\boldsymbol{\Phi}$ matrix of common factor variances and covariances have been substituted by (35) and (36). However, although $\boldsymbol{\kappa}$ does not feature in the AR3MM, hypotheses about the common factor means can still be tested by imposing restrictions on the innovation factor means in $\boldsymbol{\alpha}$. In the AR3MM, the hypothesis of equal common factor means can be investigated by imposing

$$\boldsymbol{\alpha} = (\mathbf{I} - \mathbf{B})\,(\mathbf{u} \otimes \boldsymbol{\kappa}_{\mathrm{F}}), \tag{41}$$

and the hypothesis of a linear mean trend can be investigated by imposing

$$\boldsymbol{\alpha} = (\mathbf{I} - \mathbf{B})\,(\mathbf{u} \otimes \mathbf{a} + \mathbf{t} \otimes \mathbf{b}). \tag{42}$$

Equations (41) and (42) are the result of rewriting (35) in terms of $\boldsymbol{\alpha}$ and substituting $\boldsymbol{\kappa}$ by (25) and (26). The same could be done to test the L3MM hypotheses about the common factor variances and covariances in $\boldsymbol{\Phi}$. However, rewriting (36) in terms of $\boldsymbol{\Psi}$, and restricting $\boldsymbol{\Phi}$ according to the hypothesis to be tested, results in symmetric $\boldsymbol{\Psi}$ matrices that do not conform to the assumption that innovation factors are not correlated across occasions. The hypothesis of equal common factor variances across occasions can nevertheless be investigated, but only by adding (22) as a separate constraint under which the AR3MM parameters are estimated.

Hypotheses about the residual factors can be tested within the AR3MM in the same way as has been described for the L3MM (equations (27)–(29)). Thus far we have assumed that residual factors of the same variables are merely correlated across occasions. However, analogously to (36), we could assume an autoregressive structure for the residual variances and covariances,

$$\mathrm{Cov}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}') = \boldsymbol{\Theta} = (\mathbf{I} - \mathbf{B}_{\varepsilon})^{-1}\boldsymbol{\Psi}_{\varepsilon}(\mathbf{I} - \mathbf{B}'_{\varepsilon})^{-1}, \tag{43}$$

where $\mathbf{B}_{\varepsilon}$ and $\boldsymbol{\Psi}_{\varepsilon}$ are partitioned matrices. $\mathbf{B}_{\varepsilon}$ is a square $JK \times JK$ matrix, consisting of diagonal $\mathbf{B}_{\varepsilon,j,j-1}$ matrices and zero matrices, all $K \times K$, in a structure similar to the one given by (34). $\boldsymbol{\Psi}_{\varepsilon}$ is a diagonal $JK \times JK$ matrix, consisting of J diagonal $K \times K$ matrices. An autoregressive model for the means of the residual factors is not applicable because of the assumption that the residual factor means are zero (compare (6)).

5. Latent curve three-mode models

An LC3MM is an L3MM with restrictions on $\boldsymbol{\kappa}$ and $\boldsymbol{\Phi}$ that conform to a latent curve structure for the means and covariances of the common factors (equation (49) and (50) below).

The topic of latent (growth) curve modelling has recently become very popular—see, for example, Willet and Sayer (1994) or Duncan *et al.* (1999), the latter providing a comprehensive introduction to latent curve modelling. Here we build on the work of Browne (1993), who himself acknowledges Meredith and Tisak (1990) as the originators of the latent curve analysis as we use it. Most researchers only consider linear or quadratic curves, but Browne shows how any curve can be used in a latent curve model. Although Browne only models the repeated measurement of a single observed variable, his work is easily extended to modelling mean and covariance structures of multiple latent variables.

5.1. *Latent curve model for the common factor scores*

Suppose the J scores on common factor r can be described by a target function, $f_r\,(t_j,\,\mathbf{c}_r)$, where t_j is a code for the time of occasion j (e.g., $t_j = j$), and $\mathbf{c}_r$ is an M_r-vector of function parameters. We assume that

$$E(\boldsymbol{\xi}_r) = \boldsymbol{\kappa}_r = f_r(t_j, \mathbf{c}_r), \tag{44}$$

and that all elements of the target function are differentiable with respect to $\mathbf{c}_r$. Browne (1993) gives a detailed description of the development of latent curve models for observed scores. These models are for repeated observations of a single observed variable. Here we summarize Browne's exposition, at the same time transforming it to the case of repeated measurements of multiple latent variables. Following Browne, the latent curve model for the scores of an arbitrary subject on common factor r is written as

$$\boldsymbol{\xi}_r = \mathbf{P}_r\,\boldsymbol{\eta}_r + \boldsymbol{v}_r, \tag{45}$$

where $\boldsymbol{\xi}_r$ is a J-vector of the scores on common factor r on J occasions, $\mathbf{P}_r$ is a $J \times M_r$ matrix of coefficients dependent of the target function, and $\boldsymbol{\eta}_r$ is a random M_r-vector of random coefficients, sometimes interpreted as learning factors or growth factors. We will refer to the $\boldsymbol{\eta}_r$ factors as curve factors. $\boldsymbol{v}_r$ is a random J-vector of residual factors, representing deviations of the latent curve, possibly because of measurement error. The $\mathbf{P}_r$ elements, ρ_{rjm}, are given by so-called 'basis functions', which are the partial derivatives of the target function with respect to the function parameters $\mathbf{c}_r$ (terminology after Meredith & Tisak, 1990):

$$\rho_{rjm} = g_{rm}(t_j, \mathbf{c}_r) = f_r'(t_j, \mathbf{c}_r) = \frac{\partial f_r(t_j, \mathbf{c}_r)}{\partial c_{rm}}. \tag{46}$$

Equation (45) gives the latent curve model for a single common factor r, but we have R common factors. For R common factors, the latent curve model is

$$\boldsymbol{\xi} = \mathbf{P}\boldsymbol{\eta} + \boldsymbol{v}, \tag{47}$$

where the random vectors $\boldsymbol{\xi}$ and $\boldsymbol{v}$ are partitioned JR-vectors consisting of stacked $\boldsymbol{\xi}_j$ and $\boldsymbol{v}_j$ vectors. The $\boldsymbol{\eta}$ vector consists of stacked $\boldsymbol{\eta}_r$ vectors. The length of the $\boldsymbol{\eta}$ vector is $M_1 + M_2 + \cdots + M_R$. Because of the way the $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ vectors are organized, the structure of the partitioned $\mathbf{P}$ matrix is a little complicated. The $\mathbf{P}$ matrix is $JR \times (M_1 + M_2 + \cdots + M_R)$ and consists of stacked $\mathbf{P}_j$ matrices. A $\mathbf{P}_j$ matrix is $R \times (M_1 + M_2 + \cdots + M_R)$. For example, with $R = 3$, a $\mathbf{P}_j$ matrix has the form

$$\mathbf{P}_j = \begin{vmatrix} \rho_{1j1} & \cdots & \rho_{1jM(1)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \rho_{2j1} & \cdots & \rho_{2jM(2)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \rho_{3j1} & \cdots & \rho_{3jM(3)} \end{vmatrix}, \quad (48)$$

where ρ_{rjm} are obtained through the basis functions of (46). Assuming that the residual factors $\boldsymbol{\upsilon}$ have zero means, and are correlated neither with each other nor with the curve factors $\boldsymbol{\eta}$, it follows that the means and covariances of the common factors $\boldsymbol{\xi}$ are given by

$$E(\boldsymbol{\xi}) = \boldsymbol{\kappa} = \mathbf{P}\boldsymbol{\nu}, \quad (49)$$

$$\mathrm{Cov}(\boldsymbol{\xi}, \boldsymbol{\xi}') = \boldsymbol{\Phi} = \mathbf{P\Pi P}' + \boldsymbol{\Omega}, \quad (50)$$

where $\boldsymbol{\nu}$ is a $(M_1 + M_2 + \cdots + M_R)$ vector of curve factor means, $\boldsymbol{\Pi}$ is a symmetric matrix containing the variances and covariances of the curve factors, and $\boldsymbol{\Omega}$ is a $JR \times JR$ diagonal matrix containing the variances of the residual factors $\boldsymbol{\upsilon}$. As both $\mathbf{P}$ and $\boldsymbol{\kappa}$ are fully dependent on the latent curve function parameters (equations (44) and (46), it follows from (49) that $\boldsymbol{\nu}$ can be worked out from these parameters as well, and does not contain any free parameters either. When fitting a LC3MM this can be taken care of by adding

$$f_r(\mathbf{t}, \mathbf{c}_r) = \mathbf{P}_r\boldsymbol{\nu}_r \quad (51)$$

as a separate constraint under which the LC3MM parameters are estimated (where f_r $(\mathbf{t}, \mathbf{c}_r)$ is a J-vector of f_r $(t_j, \mathbf{c}_r)$ values).

Since the LC3MM is the result of substitution of equations (49) and (50) into (19) and (20), it is clear that the LC3MM is an ordinary second-order factor model, albeit with certain restrictions on the parameter matrices.

5.2. LC3MM examples

If we assume that the subject scores on common factor k on different occasions lie on a linear curve, then the target function is

$$f_r(t_j, \mathbf{c}_r) = c_{r1} + c_{r2}t_j, \quad (52)$$

where c_{r1} and c_{r2} are the intercept and slope parameters. Following Browne's procedure, summarized above, we obtain an LC3MM with

$$\mathbf{P}_r = \begin{vmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_J \end{vmatrix} \quad (53)$$

and

$$\boldsymbol{\nu}_r = \begin{vmatrix} c_{r1} \\ c_{r2} \end{vmatrix}. \quad (54)$$

Note that, in order to build the complete $\mathbf{P}$ matrix, the $\mathbf{P}_r$ matrices must first be transformed to $\mathbf{P_j}$ matrices in the way shown by equation (48). With a quadratic curve, the target function is

$$f_r(t_j, \mathbf{c}_r) = c_{r1} + c_{r2}t_j + c_{r3}t_j^2, \tag{55}$$

the $\mathbf{P}_r$ matrix is

$$\mathbf{P}_r = \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_J & t_J^2 \end{vmatrix}, \tag{56}$$

and the $\boldsymbol{\nu}_r$ vector is

$$\boldsymbol{\nu}_r = \begin{vmatrix} c_{r1} \\ c_{r2} \\ c_{r3} \end{vmatrix}. \tag{57}$$

Usually some simple time coding is imposed on $\mathbf{P}_r$, such as $t_j = j$ or $t_j = j - 1$. Sometimes a time coding is chosen that accounts for different lags of time between successive occasions. $\mathbf{P}$ then contains fixed parameters only. However, since the LC3MM is a special case of a second-order factor model, it is clear that the t_j can be estimated. To identify the LC3MM, only one element in each $\mathbf{P}$ column has to be fixed at a non-zero value. In this way it is possible to check whether all

$$t_j > t_{j'} \quad \text{if} \quad j > j', \tag{58}$$

for $j, j' = 1, \ldots, J$, and $j \neq j'$. If this does not hold, then apparently an inappropriate curve has been chosen to model the subjects' common factor scores.

Latent curve modelling with linear and quadratic curves is well known, and we do not really need Browne's procedure to work out the $\mathbf{P}$ and $\boldsymbol{\nu}$ matrices. We therefore give yet another example, presenting a target function that perhaps is appropriate for the data set that will be used in our illustrative example below. This target function describes 'bell-shaped' curves

$$f_r(t_j, \mathbf{c}_r) = c_{r1} + c_{r4}\exp\left(-\frac{(t_j - c_{r2})^2}{c_{r3}^2}\right), \tag{59}$$

where c_{r1} and c_{r2} are parameters for the vertical and horizontal location of the curve, and c_{r3} and c_{r4} are parameters for the width and height of the curve (with $c_{r1} = 0$, $c_{r3} = s\sqrt{2}$, and $c_{r4} = (s\sqrt{(2\pi)})^{-1}$, (59) equals the density function of the normal distribution with mean c_{r2} and standard deviation s). With this target function, the basis functions are

$$g_{r1}(t_j, \mathbf{c}_r) = \frac{\partial f_r(t_j, \mathbf{c}_r)}{\partial c_{r1}} = 1, \tag{60}$$

$$g_{r2}(t_j, \mathbf{c}_r) = \frac{\partial f_r(t_j, \mathbf{c}_r)}{\partial c_{r2}} = \frac{2c_{r4}(t_j - c_{r2})}{c_{r3}^2}\exp\left(-\frac{(t_j - c_{r2})^2}{c_{r3}^2}\right), \tag{61}$$

$$g_{r3}(t_j, \mathbf{c}_r) = \frac{\partial f_r(t_j, \mathbf{c}_r)}{\partial c_{r3}} = \frac{2c_{r4}(t_j - c_{r2})^2}{c_{r3}^3} \exp\left(-1\frac{(t_j - c_{r2})^2}{c_{r3}^2}\right), \tag{62}$$

$$g_{r4}(t_j, \mathbf{c}_r) = \frac{\partial f_r(t_j, \mathbf{c}_r)}{\partial c_{r4}} = \exp\left(-\frac{(t_j - c_{r2})^2}{c_{r3}^2}\right), \tag{63}$$

and $\mathbf{P}_r$ is a $J \times 4$ matrix,

$$\mathbf{P}_r = \begin{vmatrix} g_{r1}(t_1, \mathbf{c}_r) & g_{r2}(t_1, \mathbf{c}_r) & g_{r3}(t_1, \mathbf{c}_r) & g_{r4}(t_1, \mathbf{c}_r) \\ g_{r1}(t_2, \mathbf{c}_r) & g_{r2}(t_2, \mathbf{c}_r) & g_{r3}(t_2, \mathbf{c}_r) & g_{r4}(t_2, \mathbf{c}_r) \\ \vdots & \vdots & \vdots & \vdots \\ g_{r1}(t_J, \mathbf{c}_r) & g_{r2}(t_J, \mathbf{c}_r) & g_{r3}(t_J, \mathbf{c}_r) & g_{r4}(t_J, \mathbf{c}_r) \end{vmatrix}. \tag{64}$$

From (44), (46) and (49) it follows that $\boldsymbol{\nu}_r$ does not contain free parameters. However, with the target function of (59), $\boldsymbol{\nu}_r$ does not have a clear interpretation like the $\boldsymbol{\nu}_r$ associated with the linear and quadratic curves ((54) and (57)). Still, $\boldsymbol{\nu}_r$ can be solved by adding (51) as an additional constraint to the optimization of the fitting function when estimating the LC3MM parameters (for example, by using the constraint facility of the computer program Mx (Neale *et al.*, 1999).

Note that there are an infinite number of target functions that can be used in the LC3MM. Equations (52), (55) and (59) are just three examples. For some other examples, see Browne (1993), who gives the target and basis functions for exponential, logistic and Gompertz curves.

5.3. *Hypothesis testing with LC3MMs*

Hypotheses that are specific to the LC3MM involve the target functions that make up $\mathbf{P}_r$ and $\boldsymbol{\nu}_r$. Which hypotheses can be tested, and how, depends on the target functions that are used in the LC3MM. For example, with the simple linear curve of (52), the hypothesis of equal common factor means across occasions is investigated by fixing the mean of the η_{r2} factor at zero. Note that this does not mean that the individuals do not change over time, only that on average there is no change.

If we want to test whether there are indeed individual differences between subjects on a particular curve factor, then we have to fix the associated variances and covariances in $\boldsymbol{\Pi}$. For example, with the linear curve of (52), we can test whether the subjects have different starting levels by fixing the variances (and covariances) of the η_{r1} factor at zero (or by removing that curve factor altogether).

We now turn to hypotheses about the residual factors in $\boldsymbol{\upsilon}$ in (47). Following the mainstream of the literature on the topic, we have assumed that all residual factors in $\boldsymbol{\upsilon}$ are uncorrelated, so that the $JR \times JR$ covariance matrix $\boldsymbol{\Omega}$ in (50) is diagonal. However, there is no reason why the same residual factors should not be correlated across occasions, so that $\boldsymbol{\Omega}$ has a structure of diagonal $R \times R$ blocks, just like the $JK \times JK$ covariance matrix $\boldsymbol{\Theta}$ of residual factors $\boldsymbol{\varepsilon}$, which consists of diagonal $K \times K$ blocks (see the text below equation (8)). Browne (1993) suggests an autoregressive structure for the covariance matrix of the $\boldsymbol{\upsilon}$ factors. So, analogously to (36) and (43), $\mathbf{\Omega}$ can be subject to the restriction

$$\mathrm{Cov}(\boldsymbol{v}, \boldsymbol{v}') = \mathbf{\Omega} = (\mathbf{I} - \mathbf{B}_{v})^{-1}\mathbf{\Psi}_{v}(\mathbf{I} - \mathbf{B}_{v}')^{-1}, \qquad (65)$$

where $\mathbf{B}_{v}$ and $\mathbf{\Psi}_{v}$ are partitioned $JR \times JR$ matrices, with the same structures as described for the $JK \times JK$ $\mathbf{B}_{v}$ and $\mathbf{\Psi}_{\varepsilon}$ matrices (see the text below (43)).

6. Illustration

As an example, L3MMs are applied to data from the field of industrial psychology. We use a small part of the data collected within the framework of an extensive study of work attitude in turnover situations by Frese and his co-workers (see Frese, Garst & Fay, 1998, and the references therein). This study is known as the AHUS study, after the German acronym for 'Active Actions in a Radical Change Situation'. The part of the AHUS study that we are concerned with involves the development of personal initiative and some of its predictors. Among other things, Frese *et al.* (1998) wanted to know to what extent people from East Germany showed initiative in finding a (better) job after German reunification in July 1990. In Frese's theory of occupational socialization, the effects of various work characteristics on personal initiative are mediated by various control cognitions. Here, in our limited example, we will consider the effects of just two variables, control aspiration and self-efficacy, on the showing of initiative.

6.1. Data and method

The part of the sample that is used here consists of 271 participants, 146 men and 125 women. 'Personal initiative' (PINI), 'control aspiration' (CASP) and 'self-efficacy' (SEFF) were measured in 1990, 1991, 1992 and 1993. There are three observed indicator variables for PINI, two for CASP, and three for SEFF (indicator variables were created through item parcelling). So we have eight different observed variables ($K = 8$), three different common factors ($R = 3$), all measured on four occasions ($J = 4$). We consequently have 32 (4×8) observed variables and 12 (4×3) common factors.

The scores on the 32 observed variables are not multivariate normally distributed. Univariate tests reveal that the distributions are skewed for almost half of the variables, and platykurtic for four variables. Tests of multivariate normality turn out highly significant (Mardia, 1980; skewness = 174.3, $Z = 15.7$; kurtosis = 1175.6, $Z = 11.0$; relative multivariate kurtosis = 1.08).

Non-normality poses a problem for choosing a suitable estimation method. We cannot use the weighted least squares estimation method (Bollen, 1989) because our sample size of 271 is too small. We could retain the ML estimation method, but use a corrected (scaled) test statistic (Bentler & Dudgeon, 1996; Yuan & Bentler, 1998), which alleviates the problem of non-normality somewhat. This is, however, not an option in the current version of Mx (version 1.50; Neale *et al.*, 1999), that we use for fitting L3MMs. We therefore retain the normal-theory ML estimation method and report the accompanying test statistic. As the assumption of multivariate normality is violated, the resulting test statistic need not have a chi-square distribution, and the standard errors need not be correct. However, the point estimates of model parameters are probably not seriously biased (Bollen, 1989).

Thus all the models discussed below are fitted using the normal-theory ML estimation method (based on the discrepancy function of equation (18)). In Table 1 we report three measures of fit: the chi-square measure of overall goodness of fit (CHISQ), the root mean square error of approximation (RMSEA), and the expected cross-validation index (ECVI). Browne & Cudeck (1992) provide formulae to calculate confidence intervals for both the RMSEA and the ECVI. These calculations can be done with freely available computer programs such as FITMOD.EXE of M. W. Browne (http://quantrm2.psy.ohio-state.edu/browne, included in the MUTMUM.ZIP package) and RMSEA.EXE of P. Dudgeon (http://www.mhri.edu.au/~pld). Although we may not use the chi-square distribution to interpret the (non-central) chi-square statistic, and although the derived statistics RMSEA and ECVI depend on normality and assumptions as well, we can still use RMSEA and ECVI to compare the fit of different models to the same data. Simulation studies (Curran, West & Finch, 1996) suggest that under non-normalilty, our fit measures will turn out too high rather than too low.

It is, of course, well known that no single measure of overall fit should be relied on exclusively. Moreover, the fit of the components of a model should also be evaluated (Bollen & Long, 1993). When a model does not fit sufficiently well, standardized discrepancies between the observed and estimated means, variances and covariances may give valuable information about non-trivial further structure (e.g. Jöreskog, 1993). Notwithstanding these cautionary remarks, our prime purpose is not to find a 'best' model for the AHUS data, but to illustrate various modelling aspects. To that end a large number of models are fitted, and to ease our presentation we limit our consideration of fit to the RMSEA.

6.2. *Testing measurement invariance*

First consider the L3MM with the mean and covariance structures of (19) and (20). We fit the L3MM with $K = 8, R = 3$ and $J = 4$. So the $\mathbf{\Lambda}_0$ matrix is 8×3, and the $\boldsymbol{\tau}_0$ vector is 8×1. Scales and origins for the common factors are provided by setting

$$\lambda_{0(11)} = \lambda_{0(42)} = \lambda_{0(63)} = 1, \tag{66}$$

$$\tau_{0(1)} = \tau_{0(4)} = \tau_{0(6)} = 0. \tag{67}$$

The actual patterns of fixed and free parameters in $\mathbf{\Lambda}_0$ and $\boldsymbol{\tau}_0$ can be gathered from Table 2. The parameters in the 12×12 symmetrix $\mathbf{\Phi}$ matrix and the 12×1 $\boldsymbol{\kappa}$ vector are all free to be estimated, and the 32×32 symmetric $\mathbf{\Theta}$ matrix has 80 free parameters, as all covariances between residual factors of the same variables are free to be estimated (that is, the partitioned $\mathbf{\Theta}$ matrix is made up of diagonal $K \times K$ blocks, as the residual factors of the same variables are allowed to correlate across occasions). The total number of parameters to be estimated is 180 and the sample size is 271.

Although the chi-square goodness of fit test is significant, the fit of the L3MM appears good according to RMSEA (CHISQ (380) = 546.6, RMSEA = 0.040; the L3MM is model 1.4 in Table 1). The resulting parameter estimates are given in Table 2.

Browne & Cudeck (1992) explain why the chi-square measure is not always appropriate to evaluate the fit of structural equation models. They prefer using indices like the RMSEA and the ECVI, and have provided the following rule of thumb: RMSEA values smaller than 0.05 are indicative of close fit, but values smaller than 0.08 are still considered reasonable. So the fit of the L3MM may be considered 'close'. To allow a better appreciation of the fit of the L3MM, we also fit model 1.1 which is identical to the L3MM except for the measurement

Table 1. Fit results

Model		df	CHISQ	RMSEA	ECVI
Measurement invariance					
1.1	no restrictions	350	455.0	0.033 (0.022–0.043)	3.241 (3.018–3.504)
1.2	$\mathbf{\Lambda}_j$ invariant	365	466.3	0.032 (0.020–0.042)	3.171 (2.947–3.436)
1.3	$\boldsymbol{\tau}_j$ invariant	365	466.0	0.032 (0.020–0.042)	3.171 (2.946–3.435)
1.4	L3MM: $\mathbf{\Lambda}_j$, $\boldsymbol{\tau}_j$ invariant	380	546.6	0.040 (0.031–0.049)	3.358 (3.103–3.652)
L3MMs					
2.1	equal $\boldsymbol{\xi}$ variances	389	561.4	0.041 (0.031–0.049)	3.346 (3.088–3.644)
2.2	equal $\boldsymbol{\xi}$ correlations	440	775.5	0.053 (0.046–0.060)	3.761 (3.438–4.123)
2.3	equal $\boldsymbol{\xi}$ covariances	449	792.6	0.053 (0.046–0.060)	3.758 (3.431–4.123)
2.4	equal $\boldsymbol{\varepsilon}$ variances	404	626.8	0.045 (0.037–0.053)	3.477 (3.197–3.796)
2.5	equal $\boldsymbol{\varepsilon}$ correlations	425	651.6	0.044 (0.036–0.052)	3.413 (3.129–3.737)
2.6	equal $\boldsymbol{\varepsilon}$ covariances	449	733.7	0.049 (0.041–0.056)	3.540 (3.231–3.887)
2.7	equal $\boldsymbol{\xi}$ means	389	608.4	0.046 (0.037–0.054)	3.520 (3.244–3.835)
2.8	linear trend $\boldsymbol{\xi}$ means	386	595.4	0.045 (0.036–0.053)	3.494 (3.222–3.805)
2.9	2.8, estimated time coding	384	558.6	0.041 (0.032–0.050)	3.373 (3.114–3.670)
2.10	equal $\boldsymbol{\xi}$ means and variances	398	624.8	0.046 (0.038–0.054)	3.514 (3.234–3.834)
AR3MMs					
3.1	AR1	434	831.7	0.058 (0.051–0.065)	4.014 (3.673–4.393)
3.2	AR1, SC	425	800.1	0.057 (0.050–0.064)	3.963 (3.631–4.334)
3.3	AR1, SE	425	751.7	0.053 (0.046–0.061)	3.784 (3.466–4.141)
3.4	AR1, CLE	425	761.3	0.054 (0.047–0.062)	3.820 (3.499–4.180)
3.5	AR1, AR2	428	689.7	0.048 (0.040–0.055)	3.532 (3.235–3.868)
3.6	AR1, AR2, SC	419	664.4	0.047 (0.038–0.054)	3.505 (3.215–3.835)
3.7	AR1, AR2, SE	419	628.6	0.043 (0.035–0.051)	3.373 (3.095–3.690)
3.8	AR1, AR2, CLE	419	638.4	0.044 (0.036–0.052)	3.409 (3.128–3.729)

Table 1. continued

Model		df	CHISQ	RMSEA	ECVI
4.1	3.8 with equal AR1, AR2, CLE	434	688.7	0.047 (0.039–0.054)	3.484 (3.188–3.819)
4.2	3.8 with equal ζ_j variances	425	658.6	0.045 (0.037–0.053)	3.439 (3.152–3.766)
4.3	3.8 with equal ζ_j means	425	643.1	0.044 (0.035–0.052)	3.382 (3.100–3.703)
4.4	3.8, equal ζ_j var's and means	431	662.2	0.045 (0.036–0.052)	3.408 (3.121–3.735)
4.5	4.1 with equal ζ_j var's, means	446	821.3	0.056 (0.049–0.063)	3.886 (3.550–4.260)
4.6	3.8, linear trend ξ_j means	425	692.3	0.048 (0.040–0.056)	3.564 (3.266–3.902)
4.7	3.8, equal ξ_j means	428	705.2	0.049 (0.041–0.057)	3.590 (3.288–3.931)
4.8	3.8, equal ξ_j variances	428	652.3	0.044 (0.036–0.052)	3.394 (3.109–3.717)
4.9	3.8, equal ξ_j means and var's	437	720.5	0.049 (0.041–0.057)	3.580 (3.274–3.925)
4.10	3.8 with AR1 in $\mathbf{\Theta}$ structure	443	724.0	0.049 (0.041–0.056)	3.548 (3.242–3.893)
4.11	3.8 with AR1, AR2 in $\mathbf{\Theta}$	427	671.6	0.046 (0.038–0.054)	3.473 (3.181–3.803)
4.12	4.11 with equal AR1, AR2 in $\mathbf{\Theta}$	451	699.4	0.045 (0.037–0.053)	3.398 (3.101–3.734)
LC3MMs					
5.1	linear trend for all	431	655.7	0.045 (0.037–0.053)	3.421 (3.133–3.749)
5.2	stationary trend for all	449	727.8	0.048 (0.040–0.055)	3.518 (3.133–3.749)
5.3	quadratic trend for $r = 1$	423	616.0	0.041 (0.032–0.049)	3.296 (3.024–3.608)
5.4	bell trend for $r = 1$	416	618.2	0.042 (0.034–0.051)	3.356 (3.082–3.670)
5.5	Model 5.1 with AR1 struct. $\mathbf{\Omega}$	422	651.0	0.045 (0.037–0.053)	3.433 (3.148–3.757)
5.6	Model 5.2 with AR1 struct. $\mathbf{\Omega}$	440	700.6	0.047 (0.039–0.054)	3.484 (3.185–3.821)
5.7	Model 5.3 with AR1 struct. $\mathbf{\Omega}$	414	604.4	0.041 (0.032–0.050)	3.320 (3.050–3.629)
5.8	Model 5.2 with diag. block $\mathbf{\Omega}$	431	690.9	0.047 (0.039–0.055)	3.514 (3.217–3.850)

Note: $N = 271$; RMSEA and ECVI values in parentheses denote 95% confidence intervals; AR1 = first-order autoregressive effects, AR2 = second-order autoregressive effects, SC = synchronous correlations, SE = synchronous effects, CLE = cross-lagged effects.

Table 2. L3MM parameter estimates (model 1.4)

	PINI	CASP	SEFF		$\boldsymbol{\tau}_0$
$\boldsymbol{\Lambda}_0 =$	*1.00*	*0.00*	*0.00*	$\boldsymbol{\tau}_0 =$	*0.00*
	1.04	*0.00*	*0.00*		−0.07
	1.01	*0.00*	*0.00*		−0.27
	0.00	*1.00*	*0.00*		*0.00*
	0.00	0.87	*0.00*		0.47
	0.00	*0.00*	*1.00*		*0.00*
	0.00	*0.00*	0.88		0.66
	0.00	*0.00*	1.06		−0.17

	Occasion 1			Occasion 2			Occasion 3			Occasion 4		
	PINI	CASP	SEFF	PINI	CASP	SEFF	PINI	CASP	SEFF	PINI	CASP	SEFF
$\boldsymbol{\Phi} =$	0.79											
	0.14	0.20										
	0.19	0.11	0.36									
	0.38	0.11	0.13	0.64								
	0.10	0.17	0.14	0.07	0.20							
	0.16	0.12	0.30	0.13	0.14	0.38						
	0.38	0.14	0.20	0.26	0.11	0.19	0.76					
	0.10	0.15	0.13	0.09	0.18	0.14	0.11	0.22				
	0.18	0.10	0.29	0.16	0.15	0.29	0.21	0.15	0.41			
	0.37	0.07	0.18	0.34	0.05	0.14	0.34	0.07	0.20	0.66		
	0.15	0.18	0.14	0.13	0.17	0.16	0.16	0.20	0.17	0.11	0.29	
	0.14	0.13	0.30	0.13	0.13	0.31	0.18	0.16	0.34	0.20	0.20	0.40
$\boldsymbol{\kappa}' =$	3.48	3.45	3.85	3.73	3.49	3.84	3.91	3.55	3.81	3.56	3.57	3.85

Note: $N = 271$, CHISQ (380) = 546.6, RMSEA = 0.040; other fit indices for the L3MM (model 1.4) are given in Table 1; coefficients in italics are fixed parameters; estimates of the $\boldsymbol{\Theta}$ parameters are omitted from the table to save space.

invariance constraints of equations (9) and (10), and 18 additional identification restrictions in $\boldsymbol{\Lambda}$ and $\boldsymbol{\tau}$. This model, with 210 free parameters, fits even better: CHISQ (350) = 455.0 and RMSEA = 0.033 (Table 1).

Comparison of the fit of model 1.1 and the L3MM provides a test of measurement invariance. If the more restricted L3MM fits significantly worse than model 1.1 then the measurement invariance hypothesis must be rejected. According to the difference in chi-square values the hypothesis must be rejected (CHISQ (30) = 91.6). However, the difference in chi-square values of two hierarchically related models itself has a chi-square distribution only if at least the less restricted model does fit. But according to the chi-square goodness of fit test model 1.1 does not fit either. Therefore, we compare the fit of models by calculating the confidence intervals of the RMSEA values. For model 1.1 the 95% confidence interval of the RMSEA ranges from 0.022 to 0.043. As the point estimate of the RMSEA of the L3MM falls well within the confidence interval of the RMSEA of model 1.1, we do not reject the hypothesis of measurement invariance.

We will use this L3MM as the reference model in further model comparisons below. This means that, if a hypothesis is tested by fitting a particular model that is a special case of the L3MM, then this hypotheses will be rejected only if the point estimate of the RMSEA of this special case does not fall within the confidence interval of the RMSEA of the L3MM (0.031–0.049; see model 1.4 in Table 1).

6.3. Testing L3MM hypotheses

Restricting the L3MM according to (22)–(24) tests the hypotheses about the common factor variances, correlations and covariances. These restrictions have been imposed, one at a time, in models 2.1, 2.2 and 2.3. From the fit results in Table 1 it appears that the common factor variances may be invariant across occasions (because the restricted model 2.1 does not fit significantly worse than the L3MM), but the other hypotheses must be rejected. With models 2.4, 2.5 and 2.6, the same hypotheses are tested for the residual factors as well. For these even the hypothesis of equal variances and covariances across occasions ((29)) is not clearly rejected (see Table 1).

Hypotheses about the common factor means are investigated with models 2.7, 2.8 and 2.9. In model 2.7 the common factor means are restricted according to (25), testing the hypothesis of equal common factor means across occasions. Comparison of the fit results of model 2.7 and the L3MM shows that this hypothesis cannot be rejected. The invariant common factor means of PINI, CASP and SEFF are estimated at 3.69, 3.49 and 3.81, respectively, which values are indeed not far off the unrestricted factor means that are estimated under the L3MM without the restriction of (25) (see Table 2). Nevertheless, for the purpose of illustration, we test the hypothesis of a linear trend for the common factor means as well. In model 2.8 the restriction of (26) is imposed on the $\boldsymbol{\kappa}$ vector, yielding intercept and slope estimates of

$$\mathbf{a} = \begin{vmatrix} 3.66 \\ 3.43 \\ 3.84 \end{vmatrix} \quad \text{and} \quad \mathbf{b} = \begin{vmatrix} 0.03 \\ 0.04 \\ 0.00 \end{vmatrix}. \tag{68}$$

Note that the slope parameters are almost zero, which is not surprising knowing that the hypothesis of invariant common factor means could not be rejected. However, inspection of the unrestricted common factor means of the L3MM in Table 2 shows that at least the PINI factor means do vary somewhat across time. If we fit the same model once more but with t_2 and t_3 free to be estimated, then we obtain a better fit (model 2.9 in Table 1), but the requirement of (58) is no longer satisfied, indicating that across-occasion changes of the common factor means (of PINI at least) are not adequately described by a linear trend. These last remarks do not bear much weight as we have not even discarded model 2.7, testing the invariant means hypothesis of (25).

Finally, in model 2.10 we have imposed both (22)—invariant variances—and (25)—invariant means. As this model does not fit significantly worse than the unrestricted L3MM

(Table 1), we conclude that the PINI, CASP and SEFF factor means and variances are invariant over time. Note that this does not mean that the individuals have not changed over time; this will be investigated by testing hypotheses within the framework of LC3MMs.

6.4. *Fitting AR3MMs*

The RMSEA for the basic AR3MM (model 3.1) does not fall within the 95% confidence interval of the RMSEA for the L3MM (compare models 1.4 and 3.1 in Table 1). So the basic AR3MM with only first-order autoregressive effects must be rejected. Adding synchronous correlations (model 3.2), synchronous effects (model 3.3), cross-lagged effects (model 3.4), or second-order autoregressive effects (model 3.5) to the basic AR3MM does improve the fit, but still none of these models fits quite as well as the L3MM (Table 1). Only model 3.5 yields an RMSEA that falls within the RMSEA interval for the L3MM (model 1.4). The fit further improves if second-order autoregressive effects are combined with synchronous correlations (model 3.6), synchronous effects (model 3.7) or cross-lagged effects (model 3.8).

The synchronous effects (in models 3.3 and 3.7) and the cross-lagged effects (in models 3.4 and 3.8) are effects of SEFF on PINI, SEFF on CASP, and CASP on PINI. These effects, which do seem plausible to people familiar with the AHUS study, are estimated by allowing all coefficients above the diagonal of the $\mathbf{B}_{jj}$ (synchronous effects; $j = 1, \ldots, J$) or $\mathbf{B}_{j,j-1}$ matrices (cross-lagged effects; $j = 2, \ldots, J$) to vary freely. From Table 1 it appears that model 3.7 fits best, but not significantly better than model 3.8. As we feel that in the AHUS case cross-lagged effects are more plausible than synchronous effects, we choose model 3.8 as our reference model for the subsequent testing of hypotheses. The parameter estimates for model 3.8 are presented in Table 3.

In models 4.1–4.3 we impose the across-occasion restrictions given by (38)–(40), to test for equal autoregressive and cross-lagged effects (model 4.1), equal innovation factor variances (model 4.2) and equal innovation factor means (model 4.3), respectively. The point estimates of the RMSEA for these models all fall within the 95% confidence interval of the RMSEA for model 3.8, and these individual hypotheses are not rejected. The same goes for the hypothesis of both the means and the variances of the innovation factors being invariant across occasions, investigated by combining the restrictions of (39) and (40) (model 4.4). But if we require the autoregressive and cross-lagged effects to be invariant as well, we obtain a model that does not fit (model 4.5, Table 1).

With model 4.6 we test the hypothesis that the successive common factor means can be described by a linear trend, imposing the restriction of (42) on the $\boldsymbol{\alpha}$ vector of model 3.8. The fit of model 4.6 is not significantly worse than that of model 3.8, so this hypothesis is not rejected. The resulting estimates for the intercepts and slopes are

$$\mathbf{a} = \begin{vmatrix} 3.58 \\ 3.38 \\ 3.84 \end{vmatrix} \quad \text{and} \quad \mathbf{b} = \begin{vmatrix} 0.04 \\ 0.05 \\ 0.00 \end{vmatrix}. \tag{69}$$

These estimates hardly differ from the intercept and slope estimates for the linear trends of the common factor means in the L3MM ((68)). As with the L3MM, the slopes are close to zero, and the hypothesis of a stationary trend in the AR3MM (equation (41), or (42) with $\mathbf{b} = 0$) is not rejected either (model 4.7, Table 1).

The hypothesis of equal common factor variances is tested by comparing the fit of models 3.8 and 4.8. The specification of these two AR3MMs is the same, but the matrix $\mathbf{\Phi}$ is written as in (21) and the parameters of model 4.8 are estimated under the constraint of (22). This yields an additional nine degrees of freedom, and the relative fit of model 4.8 is as good as the fit of model 3.8 (Table 1). Restricting model 3.8 by imposing both the restrictions of (22) and (41) yields model 4.9, the fit of which is still satisfactory (Table 1). So we arrive at the same conclusion as before (with model 2.10, that is). Across occasions, the PINI, CASP and SEFF factor means and variances appear invariant.

Model 4.10 is obtained by specifying a first-order autoregressive structure for the variances and covariances of the residual factors ε ((43)) in model 3.8. The fit of model 4.10 is satisfactory, but adding second-order autoregressive effects on the residual factors still improves the fit (model 4.11; Table 1). Additional imposition of equality constraints on all (same order) autoregressive effects in $\mathbf{B}_\varepsilon$ (model 4.12) yields so many degrees of freedom, that the RMSEA decreases again (Table 1).

6.5. *Fitting LC3MMs*

LC3MMs are obtained by restricting the L3MM (model 1.4) according to latent curve models for the common factors. The first LC3MM that we fit is a model where the scores on the common factors are described by simple linear curves. So, in model 5.1, we assume the target function of (52) for all three common factors, yielding three (4×2) $\mathbf{P}_r$ matrices as in (53). The time of the occasions is coded by $t_j = j$. The three $\mathbf{P}_r$ matrices are used to build four $\mathbf{P}_j$ matrices, as shown in (48), which are stacked to make up a (12×6) $\mathbf{P}$ matrix. The (6×1) $\boldsymbol{\nu}$ vector contains the means of intercept and slope factors and is made up of three stacked $\boldsymbol{\nu}_r$ vectors are given by (54). The symmetric (6×6) $\mathbf{\Pi}$ matrix contains the variances and covariances of the intercept and slope factors, and the diagonal (12×12) $\mathbf{\Omega}$ matrix contains the variances of the residual factors $\boldsymbol{v}$.

The fit of this model is satisfactory. The point estimate of the RMSEA of model 5.1 does not fall outside the 95% confidence interval of the RMSEA of the L3MM (model 1.4, Table 1). The means of the intercept factors are 3.57, 3.41 and 3.85, and the means of the slope factors are 0.04, 0.04 and −0.00, for PINI, CASP and SEFF, respectively. These figures hardly differ from the intercepts and slopes that have been estimated for the linear curves describing the common factor means in the L3MM (equation (68)) and AR3MM (equation (69)). The variances of the intercept factors in the LC3MM are estimated at 0.41, 0.18 and 0.33, and the variances of the slope factors at 0.00, 0.01 and 0.01. So the slope factor scores hardly vary across subjects. This means that most individuals develop their PINI, CASP and SEFF at the same rate, albeit at different levels. As the means of the slope factors are close to zero as well, we can try to do away with the slope factors altogether. Model 5.2 is a LC3MM with intercept factors only, thus describing stationary trends for all three common factors. The fit of this model is not as good as that of model 5.1, but as its RMSEA point estimate does fall within the confidence interval of the RMSEA of model 5.1, we cannot reject the idea of stationary trends.

Models 5.1 and 5.2 may reflect reality, but an alternative explanation is that the common factor scores are better described by curves other than linear curves. Looking at the estimated common factor means of the L3MM (vector $\boldsymbol{\kappa}$ in Table 2) we see that the means of the PINI factor are probably better described by a parabola or a bell-shaped curve. Therefore we

Table 3. AR3MM parameter estimates (model 3.8)

$\mathbf{\Lambda}_0 =$

PINI	CASP	SEFF
1.00	0.00	0.00
1.03	0.00	0.00
1.01	0.00	0.00
0.00	1.00	0.00
0.00	0.82	0.00
0.00	0.00	1.00
0.00	0.00	0.88
0.00	0.00	1.06

$\boldsymbol{\tau}_0 =$

0.00
−0.06
−0.27
0.00
0.62
0.00
0.69
−0.16

$\mathbf{B} =$

Occasion 1			Occasion 2			Occasion 3			Occasion 4		
PINI	CASP	SEFF	PINI	CASP	SEFF	PINI	CASP	SEFF	PINI	CASP	SEFF
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.44	0.17	0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.80	0.14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.85	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.35	0.00	0.00	0.11	0.21	0.26	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.09	0.00	0.00	0.78	0.07	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.58	0.00	0.00	0.29	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.38	0.00	0.00	0.28	−0.18	0.29	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.23	0.00	0.00	0.66	0.13	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.41	0.00	0.00	0.56	0.00	0.00	0.00

fit two other LC3MMs, models 5.3 and 5.4. In model 5.3, a quadratic curve is specified for the individual factor scores on the first factor, while retaining the simple linear curves for the other two factors. This model fits almost as well as the L3MM (although not significantly better than the other LC3MMs). The parameter estimates for model 5.3 are presented in Table 4. It appears that the third curve factor for common factor PINI (that is, associated with c_{13}) has no variance. In model 5.4 we have used equations (59)–(63) to specify a bell curve for the PINI scores, but the present dataset has too few occasions to accurately estimate the means and variances of all curve factors. If we fix the second and fourth parameter in (59) ($c_{12} = 2.5$, $c_{14} = 1$) we obtain a fit for model 5.4 that is almost as good as that of model 5.3 (Table 1).

In the LC3MMs discussed above, $\mathbf{\Omega}$ matrix is a diagonal matrix containing only the variances of the residual factors $\boldsymbol{\upsilon}$. In models 5.5, 5.6 and 5.7 we have specified a first-order autoregressive structure for the variance–covariance matrix $\mathbf{\Omega}$ of models 5.1, 5.2 and 5.3 (as

Table 3. continued

	Occasion 1			Occasion 2			Occasion 3			Occasion 4		
	PINI	CASP	SEFF	PINI	CASP	SEFF	PINI	CASP	SEFF	PINI	CASP	SEFF
$\mathbf{\Psi} =$	0.79											
	0.12	0.20										
	0.19	0.12	0.36									
	0.00	*0.00*	*0.00*	0.44								
	0.00	*0.00*	*0.00*	*0.00*	0.04							
	0.00	*0.00*	*0.00*	*0.00*	*0.00*	0.12						
	0.00	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	0.53					
	0.00	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	0.06				
	0.00	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	0.16			
	0.00	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	0.39		
	0.00	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	0.09	
	0.00	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	*0.00*	0.08
$\boldsymbol{\alpha}' =$	3.48	3.44	3.85	1.23	0.21	0.58	0.58	0.27	0.48	0.55	−0.04	0.13

Note: $N = 271$, CHISQ $(419) = 638.4$, RMSEA $= 0.044$; other fit indices for this AR3MM (model 3.8) are given in Table 1; coefficients in italics are fixed parameters; estimates of the $\mathbf{\Theta}$ parameters are omitted from the table to save space.

in (65)). This does not significantly affect the fit (Table 1). Only the relative fit of model 5.7 is marginally better than the fit of model 5.2. In model 5.8 we have tried an $\mathbf{\Omega}$ structure of diagonal blocks but that did not further improve the fit (Table 1).

6.6. Conclusion

The preferred versions of the AR3MM (model 3.8) and the LC3MM (model 5.3) do not fit significantly worse than the basic L3MM (model 1.4). Some people contend that in deciding between different models, substantive reasoning should prevail over the evaluation of fit statistics anyway. Although, in the case of the AHUS study, we had no strong ideas of what structure should be present in the data beforehand, we do have some reservations about the AR3MMs, as we do not like the fact that only AR3MMs that include second-order autoregressive effects show satisfactory fit (Table 1). We do not see why these postponed effects should apply to the yearly measurement of PINI, CASP and SEFF, and we note that data of just four occasions are hardly summarized by a second-order autoregressive model.

We have some reservations about the LC3MMs as well. From the parameter estimates for model 5.3 (Table 4) it appears that some of the curve factor variances are very small. This may indicate that a latent curve model does not apply to the AHUS data. Or perhaps other curves than the ones considered here are more appropriate for the AHUS data. We have already seen that model 5.2 with stationary curves for all three AHUS dimensions shows reasonable fit as well.

Table 4. LC3MM parameter estimates (model 5.3)

$$
\begin{array}{l}
\qquad\quad \text{PINI} \quad \text{CASP} \quad \text{SEFF} \\
\boldsymbol{\Lambda}_0 = \left|\begin{array}{ccc}
\mathit{1.00} & \mathit{0.00} & \mathit{0.00} \\
1.04 & \mathit{0.00} & \mathit{0.00} \\
1.01 & \mathit{0.00} & \mathit{0.00} \\
\mathit{0.00} & \mathit{1.00} & \mathit{0.00} \\
\mathit{0.00} & 0.82 & \mathit{0.00} \\
\mathit{0.00} & \mathit{0.00} & \mathit{1.00} \\
\mathit{0.00} & \mathit{0.00} & 0.88 \\
\mathit{0.00} & \mathit{0.00} & 1.05
\end{array}\right|
\qquad
\boldsymbol{\tau}_0 = \left|\begin{array}{r}
\mathit{0.00} \\ -0.06 \\ -0.26 \\ \mathit{0.00} \\ 0.64 \\ \mathit{0.00} \\ 0.66 \\ -0.14
\end{array}\right|
\end{array}
$$

$$
\mathbf{P} = \left|\begin{array}{ccc|cc|cc}
\multicolumn{3}{c|}{\text{PINI}} & \multicolumn{2}{c|}{\text{CASP}} & \multicolumn{2}{c}{\text{SEFF}} \\
\eta_{11} & \eta_{12} & \eta_{13} & \eta_{21} & \eta_{22} & \eta_{31} & \eta_{32} \\
\mathit{1} & \mathit{1} & \mathit{1} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{1} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{1} \\
\mathit{1} & \mathit{2} & \mathit{4} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{2} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{2} \\
\mathit{1} & \mathit{3} & \mathit{9} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{3} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{3} \\
\mathit{1} & \mathit{4} & \mathit{16} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{4} & \mathit{0} & \mathit{0} \\
\mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{0} & \mathit{1} & \mathit{4}
\end{array}\right|
\qquad
\text{vec diag}\ \boldsymbol{\Omega} = \left|\begin{array}{c}
0.19 \\ 0.03 \\ 0.05 \\ 0.39 \\ 0.03 \\ 0.09 \\ 0.47 \\ 0.04 \\ 0.10 \\ 0.28 \\ 0.07 \\ 0.02
\end{array}\right|
$$

$$
\boldsymbol{\Pi} = \left|\begin{array}{rrrrrrr}
1.361 & & & & & & \\
-0.519 & 0.149 & & & & & \\
0.075 & -0.013 & 0.000 & & & & \\
0.086 & 0.033 & -0.011 & 0.174 & & & \\
0.003 & -0.000 & 0.000 & -0.005 & 0.005 & & \\
0.236 & -0.050 & 0.008 & 0.118 & 0.002 & 0.328 & \\
-0.022 & 0.009 & -0.000 & -0.006 & 0.006 & -0.017 & 0.011
\end{array}\right|
\qquad
\boldsymbol{\nu} = \left|\begin{array}{r}
2.830 \\ 0.784 \\ -0.150 \\ 3.407 \\ 0.044 \\ 3.845 \\ -0.001
\end{array}\right|
$$

Note: $N = 271$, CHISQ (423) = 616.0, RMSEA = 0.041; other fit indices for this LC3MM (model 5.3) are given in Table 1; coefficients in italics are fixed parameters; vec diag $\boldsymbol{\Omega}$ is the diagonal of the $\boldsymbol{\Omega}$ matrix converted to a vector; estimates of the $\boldsymbol{\Theta}$ parameters are omitted from the table to save space.

In conclusion, we suggest that if there is no substantive theory about the structure of the longitudinal data at hand, then it is perhaps best to use the L3MM for the testing of substantive hypotheses. For the AHUS case this has been done with models 2.1–2.10 (Table 1). One of our examples was the hypothesis that there is no development of the means and variances of the three AHUS dimensions. This hypothesis could not be rejected (model 2.10, Table 1). The fitting of various LC3MMs shows that this conclusion also applies to the development of PINI, CASP and SEFF of individual subjects, as the estimated means and variances of the slope factors are insubstantial.

7. Discussion

The mean and covariance structures of the L3MM, given by (19) and (20), have been presented as a special case of an S3MM, which itself is a special case of the general linear latent variable model. The S3MMs are obtained by imposing one or more of the Kronecker-product constraints, given by (12)–(17), on the parameter matrices that feature in the mean and covariance structure equations of the linear latent variable model, given by (6) and (7). The L3MMs are obtained by imposing the constraints of (9) and (10), thus requiring measurement invariance across occasions.

In this way we have shown how the general theory of structural equation modelling (Bollen, 1989; Jöreskog & Sörbom, 1996) applies to the L3MMs, and that in principle L3MMs can be fitted with standard software for structural equation models. Special cases of the L3MM are obtained by further restricting the mean and covariance structures of the common factors of the L3MM. Autoregressive models and latent curve models have been given as examples. Imposition of (35) and (36) on the L3MM yields AR3MMs, and imposition of (49) and (50) yields LC3MMs.

A model for multivariate longitudinal data that does not readily fit into the framework of L3MMs is McDonald's (1984) invariant factors model (IFM). In the IFM the common factors are assumed invariant, and therefore there are just R common factors in the model, instead of JR. The IFM can be written as an S3MM where the measurement invariance restriction on the $\boldsymbol{\tau}$ vector still applies (equation (10)), but where the restriction on the $\boldsymbol{\Lambda}$ matrix now features the $\boldsymbol{\Gamma}$ matrix of (12):

$$\boldsymbol{\Lambda} = (\mathbf{I}_{J\times J} \otimes \boldsymbol{\Lambda}_0)\,\boldsymbol{\Gamma}, \tag{70}$$

where $\boldsymbol{\Gamma}$ is $JR \times R$, consisting of J transformation matrices $\boldsymbol{\Gamma}_j$, stacked vertically, each $\boldsymbol{\Gamma}_j$ being a $R \times R$ lower-triangular matrix. These $\boldsymbol{\Gamma}_j$ matrices are used to transform the basic common factor covariance matrix $\boldsymbol{\Phi}_{R\times R}$ to the particular common factor covariance matrix for occasion j. That is, in the IFM, $\boldsymbol{\Gamma}_j\,\boldsymbol{\Phi}_{R\times R}\,\boldsymbol{\Gamma}_j'$, gives the variances and covariances of the common factor scores on occasion j. Likewise, $\boldsymbol{\Gamma}_j\boldsymbol{\kappa}_{R\times 1}$ gives the means of the common factors on occasion j. In this way several hypotheses can be tested with the IFM. For example, restricting all $\boldsymbol{\Gamma}_j$ matrices to be diagonal means that the correlations between factor scores do not change across occasions, and restricting all $\boldsymbol{\Gamma}_j$ matrices to be identity ($\boldsymbol{\Gamma}_j = \mathbf{I}_{R\times R}$) means that the variances, covariances and means of the common factors do not change either. This latter hypothesis can also be tested with the L3MM, namely by restricting the $\boldsymbol{\Phi}$ matrix and the $\boldsymbol{\kappa}$ vector as in (16) and (17), with $\boldsymbol{\Phi}_S = \mathbf{U}_{J\times J}$ and $\boldsymbol{\kappa}_S = \mathbf{u}_{J\times 1}$. It is no surprise that this very restrictive version of the IFM does not fit the AHUS data (CHISQ $(461) = 2337.6$, RMSEA $= 0.123$).

Many variations of L3MMs, AR3MMs and LC3MMs have been fitted to the AHUS data. Note that in our illustration we have always investigated hypotheses by imposing restrictions on all three AHUS dimensions simultaneously. Some hypotheses are better tested for each dimension separately (as with models 5.3 and 5.4, where different curves are fitted for different AHUS dimensions). Other variations of the L3MMs not considered in our example are arrived at by first imposing a Kronecker-product restriction on one of the L3MM matrices (for example, the $\mathbf{\Phi}$ restriction of equation (16)), and subsequently imposing some structure on one of the component matrices ($\mathbf{\Phi}_S$ or $\mathbf{\Phi}_F$). Kroonenberg & Oort (1999) give an example where the Kronecker-product restriction of (15) is imposed on the residual covariance matrix $\mathbf{\Theta}$, and autoregressive structures are imposed on the $\mathbf{\Theta}_S$ component matrix. Another obvious extension of the L3MM and its special cases is to apply them to the data of multiple groups. Measurement invariance restrictions must then be imposed across groups as well (that is, $\mathbf{\Lambda}$ matrices and $\boldsymbol{\tau}$ vectors must be equal across groups), so that hypotheses concerning group differences can be tested.

In spite of some limitations, the AHUS data do serve to show various examples of hypothesis testing with L3MMs, AR3MMs and LC3MMs. The AHUS example also shows that one cannot decide between different models on the basis of fit statistics alone.

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