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FITTING THE OFF-DIAGONAL DEDICOM MODEL IN THE LEAST-SQUARES SENSE BY A GENERALIZATION OF THE HARMAN AND JONES MINRES PROCEDURE OF FACTOR ANALYSIS

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Harshman's DEDICOM model provides a framework for analyzing square but asymmetric matrices of directional relationships among n objects or persons in terms of a small number of components. One version of DEDICOM ignores the diagonal entries of the matrices. A straightforward computational solution for this model is offered in the present paper. The solution can be interpreted as a generalized Minres procedure suitable for handling asymmetric matrices.

Key words: minimum residual analysis, analysis of directional relationships.

DEcomposition into DIrectional COMponents (DEDICOM) has been suggested by Harshman (1978) as a method for the analysis of square matrices of directional data. Such matrices contain intrinsically asymmetric relations among n objects. Typical examples are friendship (interpersonal attraction) matrices, brand-switching probability matrices, or confusion matrices, see Harshman, Green, Wind and Lundy (1982).

Let X denote an n by n directional data matrix. In the DEDICOM model X is decomposed as

$$X = ARA' + E, \tag{1}$$

where A is an $n \times q$ matrix ($q < n$) of coefficients expressing the association between the n objects and q "basic types" of objects, R is a $q \times q$ matrix giving the directional relationships among the basic types, and E is an $n \times n$ matrix of error terms.

Equivalently, any element x_{ij} of X is decomposed as

$$x_{ij} = \sum_{s=1}^{q} \sum_{t=1}^{q} a_{is}a_{jt}r_{st} + e_{ij}. \tag{2}$$

It is clear from (2) that x_{ij} is approximated as the sum of q^2 triple products. Each of these products contains the association a_{is} of subject i and type s, the association a_{jt} of object j and type t, and the directional association of type s with respect to type t. As a result, a large entry x_{ij} in a friendship matrix, for instance, arises when type s strongly prefers type t and persons i and j are strongly associated with type s and t, respectively.

Harshman (1978, p. 15) noted that the diagonal entries of X are often incomparable with the off-diagonal entries in terms of the underlying psychological processes. Therefore, instead of fitting the "complete" model (1) one may often prefer to fit the off-diagonal model

$$X - \text{Diag}\,(X) = ARA' - \text{Diag}\,(ARA') + E - \text{diag}\,E. \tag{3}$$

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At this point it is instructive to compare (1) and (3) with familiar factor analytic models. Harshman et al. (1982) have pointed out that the matrix A resembles a matrix of factor loadings and that R is analogous to a matrix of correlations or covariances between oblique factors. In fact, when X is a symmetric matrix of covariances or correlations then (1) reduces to principal components analysis and (3) reduces to factor analysis.

The present paper deals with computational solutions for fitting the off-diagonal DEDICOM model in the least-squares sense. Specifically, it will be shown that (3) is not only a generalized factor analysis model conceptually, but that it can also be handled by a generalized factor analysis algorithm computationally. That is, a computational solution for (3) can be obtained by generalizing the MINRES algorithm (Harman & Jones, 1966) to handle asymmetric matrices.

A Direct Solution for the Off-Diagonal DEDICOM Model

Fitting the complete DEDICOM model (1) amounts to minimizing the function

$$f(A, R) = \|X - ARA'\|^2 = \sum_{i,j} (x_{ij} - a_i'Ra_j)^2, \tag{4}$$

where a_i' is row i of A and a_j is the transpose of row j of A.

Similarly, fitting the off-diagonal model (3) amounts to minimizing

$$g(A, R) = \|X - \text{Diag}\,(X) - ARA' + \text{Diag}\,(ARA')\|^2 = \sum_{i \neq j} (x_{ij} - a_i'Ra_j)^2. \tag{5}$$

It is obvious that $f \geq g$, with $f = g$ only if Diag (X) = Diag (ARA'). Furthermore, it is well-known that, for fixed A, the minimizing R of (4) is given by

$$R = (A'A)^{-1}A'XA(A'A)^{-1}, \tag{6}$$

if the inverse exists (Penrose, 1956). On the other hand, minimizing (4) as a function of A, for fixed R, seems quite cumbersome. Although various solutions have been suggested (Harshman et al., 1982; Harshman & Kiers, 1987) these are either approximate least-squares solutions or they lack the property of monotone convergence. Recently, Kiers (1989) developed a truly alternating least-squares method for minimizing (4), based on column-wise updating of A. The method converges monotonely, but is sometimes painfully slow.

Solutions for minimizing (5) can be obtained from methods that minimize (4), with an additional step added to each iteration. That is, after updating R and A one may replace the diagonal elements of X by their least-squares estimates, based on the current values of the model parameters, see Harshman (1981, p. 5). For instance, one might iteratively update R according to (6), update the columns of A according to the Kiers (1989) method and replace Diag (X) by Diag (ARA'). Harshman (1981, p. 5) noted that it is perhaps more elegant to eliminate the extra step and instead modify the main steps in such a way that they simply do not depend on the diagonal entries. It will now be shown that, in fact, (5) can be minimized as a function of each row of A directly, which allows us to construct a more straightforward algorithm for minimizing (5).

Specifically, let the vectors c_i and r_i contain the $n - 1$ off-diagonal elements of column i and row i of X, respectively. Then the problem of minimizing $g(A, R)$ as a function of row i of A alone amounts to minimizing

$$g_i(a_i) = \|c_i - A_iRa_i\|^2 + \|r_i - A_iR'a_i\|^2, \tag{7}$$

TABLE 1

Values of g(A,R) and CPU-times (in parentheses) for eight sets of data.

Data set	n	q	Generalized Minres	Takane	Generalized Minres	Takane
1	8	2	4594.17 (6.6)	4611.34 (4.1)	4590.99 (16)	4591.18 (16)
1	8	3	2570.85 (7.2)	2763.80 (.4)	2568.71 (16)	2657.12 (16)
1	8	4	511.58 (15.4)	1493.34 (24.4)	511.53 (16)	1523.01 (16)
2	8	2	940.70 (1.6)	940.77 (.7)	940.69 (16)	940.69 (16)
2	8	3	728.89 (12.7)	748.61 (1.2)	728.09 (16)	738.97 (16)
2	8	4	440.33 (27.7)	479.13 (14.3)	443.80 (16)	477.70 (16)
3	16	2	1302.10 (6.4)	1302.14 (2.2)	1302.10 (16)	1302.10 (16)
3	16	3	689.64 (5.9)	689.80 (5.6)	689.61 (16)	689.61 (16)
3	16	4	360.00 (48.9)	509.05 (13.9)	379.57 (16)	508.28 (16)
4	6	3	10795.59 (26.8)	11703.75 (13.5)	10932.32 (16)	11624.75 (16)
5	6	3	9226.87 (3.5)	9233.81 (1.4)	9225.67 (16)	9225.67 (16)
6	6	3	10246.69 (19.4)	11123.07 (12.9)	10269.23 (16)	11032.83 (16)
7	6	3	9828.11 (3.5)	9830.47 (1.6)	9824.54 (16)	9824.54 (16)
8	6	3	13219.17 (3.0)	13977.93 (.4)	13215.03 (16)	12894.39 (16)

where A_i is the $(n - 1) \times q$ matrix obtained by deleting row i of A. Clearly, (7) can be rearranged into

$$g_i(a_i) = \left\|\binom{c_i}{r_i} - \binom{A_iR}{A_iR'}a_i\right\|^2, \tag{8}$$

which is an ordinary least-squares regression problem, with the well-known solution

$$a_i = (R'A_i'A_iR + RA_i'A_iR')^{-1}(R'A_i'c_i + RA_i'r_i). \tag{9}$$

In the special case where X is symmetric we have $c_i = r_i$ and it can be shown that the minimizing R of (4) and (5) is also symmetric of necessity. When, additionally, the constraint $R = I$ is imposed, then updating the rows of A by (9) reduces to the Minres method of factor analysis (Harman & Jones, 1966).

We are now in a position to outline a straightforward iterative procedure for minimizing (5). After specifying q, the desired rank of the model (3), and filling A and R with arbitrary initial values, the following three steps are to be cycled through iteratively:

Step 1: Replace Diag (X) by Diag (ARA'). This yields $f(A, R) = g(A, R)$ for the current A and R.

Step 2: Replace R by the matrix that minimizes f for fixed A, according to (6). Because $g \leq f$, this step decreases $g(A, R)$ at least as much as it decreases $f(A, R)$.

Step 3: Replace each row of A in turn so as to minimize g conditionally according to (9).

Clearly, Step 1 does not affect g, but Steps 2 and 3 decrease $g(A, R)$ monotonely. We have thus obtained a monotonely convergent algorithm for minimizing (5), or, equivalently, for fitting the off-diagonal DEDICOM model (3) in the least-squares sense. The algorithm owes its simplicity to the way in which rows of A can be optimized by (9) when the diagonal elements of X can be ignored. Unfortunately, when the complete DEDICOM model (1) has to be fitted, no such simple procedure for improving A is available. The procedure offered by Kiers (1989) for updating the columns of A is far more complicated than our (9).

Practical Experiences With the Proposed Algorithm

In order to gain an impression of the efficiency of the "generalized minres approach" proposed above, it was applied to a number of data sets. For purposes of comparison, the indirect approach based on iteratively minimizing (4), with updating Diag (X) as an additional step, was also implemented. Two methods for minimizing (4) were considered, namely the method of Takane (1985) and the method of Kiers (1989). Because Takane's method appeared to converge faster than the Kiers method throughout, while on the average the same residual sums of squares (5) were obtained, it was decided to report only the results for the indirect approach based on Takane's method.

Table 1 contains the obtained values of $g(A, R)$, see (5), for three real-life data sets (1, 2 and 3), using dimensionalities 2, 3 and 4 in each case, and for five random matrices, using dimensionality 3 throughout. All iterations were started with the $n \times q$ matrix A containing the eigenvectors of $(X + X')$, associated with the largest (absolute) q eigenvalues.

On the left hand side of Table 1 the values of $g(A, R)$ and the associated CPU-times (seconds) are reported that were obtained when the iterations were terminated as soon as the difference between $g(A, R)$ and its previous value, evaluated after a full cycle of updating A and R, was smaller than .0001 times that previous value. Clearly, using this time-independent stopping criterion, it appeared that the generalized Minres procedure yields lower values of $g(A, R)$ in all 14 cases considered. However, Takane's method required far less CPU-time on the average, and stopped sooner in every case but one (data set 1, $q = 4$). Therefore, it is conceivable that the superiority of the generalized Minres approach is an artifact of the stopping criterion used. In order to verify this, the same data sets were analyzed once more, allowing a fixed CPU-time of 16 seconds. The obtained values of $g(A, R)$ after 16 seconds CPU-time are reported on the right hand side of Table 1. Although the results for Takane's method do seem to have been improved, the superiority of the generalized Minres approach still prevails. In one case Takane's method gave a lower value of $g(A, R)$ than obtained with generalized Minres, in five cases the values of $g(A, R)$ were equal, and in the remaining eight cases generalized Minres proved to be better.

From the results of Table 1 it can be concluded that the generalized Minres approach is a useful method for fitting the off-diagonal DEDICOM model.

Discussion

The algorithm suggested above fits the off-diagonal DEDICOM model while it ignores the diagonal values of ARA' altogether. As a result, these diagonal values may appear to have negative signs. Even if their sizes are irrelevant, their signs may still be a matter of concern, and one might want to impose a nonnegativity constraint on Diag (ARA'). However, such constraints seem to call for an entirely different algorithm. Minimizing (5) as a function of any single row of A is an unconstrained problem. Even if a constrained version ($a_i'Ra_i \geq 0$) of this problem could be solved, it is not at all clear how (6) could be adjusted to preserve the nonnegativity.

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