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SOME CLARIFICATIONS OF THE CANDECOMP ALGORITHM APPLIED TO INDSCAL

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Carroll and Chang have claimed that CANDECOMP applied to symmetric matrices yields equivalent coordinate matrices, as needed for INDSCAL. Although this claim has appeared to be valid for all practical purposes, it has gone without a rigorous mathematical footing. The purpose of the present paper is to clarify CANDECOMP in this respect. It is shown that equivalent coordinate matrices are not granted at global minima when the symmetric matrices are not Gramian, or when these matrices are Gramian but the solution not globally optimal.

Key words: CANDECOMP, PARAFAC, INDSCAL.

Carroll and Chang (1970) and Harshman (1970) have independently suggested the same method of analyzing three-way arrays and christened this method CANDECOMP and PARAFAC, respectively. If Z is a $p \times q \times m$ three-way array containing m frontal slabs $Z_1, \ldots, Z_m$, CANDECOMP/PARAFAC seeks to minimize the function

$$f(X, Y, D_1, \ldots, D_m) = \sum_{i=1}^{m} \|Z_i - XD_iY'\|^2, \tag{1}$$

were X is a $p \times r$ matrix, Y is a $q \times r$ matrix, D_i is a diagonal $r \times r$ matrix, and r is a fixed rank-parameter. Carroll and Chang (1970) also considered the function

$$g(X, D_1, \ldots, D_m) = \sum_{i=1}^{m} \|S_i - XD_iX'\|^2, \tag{2}$$

where, for $i = 1, \ldots, m$, S_i is a given symmetric $p \times p$ matrix, and X_i and D_i are as in (1). This function is to be minimized in the well-known INDSCAL method. To minimize (2), Carroll and Chang suggested using CANDECOMP, and justified this by claiming that, when the CANDECOMP process (applied to $S_1, \ldots, S_m$) finally converges, X and Y will be equivalent in the sense that their columns will be equal up to scalar multiplication. More recently, the claim has been repeated by Carroll and Pruzansky (1984), among others.

Ten Berge, Kiers, and de Leeuw (1988) have shown that, for a contrived set of matrices, nonequivalence may hold at certain accumulation points of the CANDECOMP/PARAFAC process. Their result, however, is not incompatible with the equivalence claim of Carroll and Chang, because f has no minimum, and hence CANDECOMP does not converge for the data set and rank ($r = 2$) they considered. Practical experience with CANDECOMP has shown that equivalence is indeed guaranteed for all practical purposes. However, mathematical proofs for equivalence have been absent.

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The purpose of the present paper is to examine equivalence from a mathematical point of view, as a further clarification of CANDECOMP applied to INDSCAL.

The organization of the present paper is as follows. First, the phenomenon of equivalence (X and Y proportional columnwise) is related to symmetry of the matrices XD_iY'. Symmetry is necessary and in most cases sufficient for equivalence; hence, examining conditions for symmetry is relevant for examining equivalence. Next, we consider the case $m = 1$ for (1) and (2), and show that asymmetry is possible in the case of certain indefinite matrices. This permits the construction of cases where CANDECOMP has asymmetric solutions, when applied to indefinite matrices $S_1, \ldots, S_m$, for $m > 1$. Finally, the case where $S_1, \ldots, S_m$ are Gramian (nonnegative definite) is treated. Surprisingly, it is shown that the CANDECOMP function does have stationary points where nonequivalence holds. On the other hand, equivalence can be shown to hold at the global minimum of the CANDECOMP function if $r = 1$ or if X and Y are constrained to be columnwise orthonormal. Neither a proof nor a counterexample to symmetry have been found for $r > 1$ and X and Y unconstrained, at the global minimum of the CANDECOMP function.

From Equivalence to Symmetry

When CANDECOMP is applied to symmetric matrices $S_1, \ldots, S_m$, upon convergence we obtain "regression" matrices

$$\hat{S}_i = XD_iY', \tag{3}$$

for $i = 1, \ldots, m$. If a CANDECOMP solution is to be of use for INDSCAL, X must equal Y, or at least X and Y should be proportional columnwise (equivalent). It is important to note that equivalence is directly related to symmetry of the regressions, henceforth referred to as "symmetry". Results 1 and 2 below pinpoint this relationship.

Result 1. Equivalence is sufficient for symmetry.

Proof. Trivial. □

Harshman (1972) has shown that, given the regression matrices $\hat{S}_1, \ldots, \hat{S}_m$, the set of matrices X, Y and $D_1, \ldots, D_m$ that satisfy (3) is unique up to certain permutations and scalar multiplications, provided that at least one pair D_i, D_j satisfies the conditions that they are nonsingular, all diagonal elements of $D_iD_j^{-1}$ are distinct, and X and Y have full column rank. The latter conditions, referred to as the "uniqueness conditions", also play a role in the next result.

Result 2. Equivalence is necessary for symmetry if the uniqueness conditions of Harshman are satisfied.

Proof. Let it be assumed that symmetry holds, and that for a pair D_i and D_j, both nonsingular, the diagonal elements of $D_iD_j^{-1}$ are distinct. Then, we have

$$XD_iY' = YD_iX', \tag{4}$$

and

$$XD_jY' = YD_jX'. \tag{5}$$

Because X and Y span the same column-space, $X = YT$ for some nonsingular matrix T, and hence,

$$TD_i = D_i T', \tag{6}$$

and

$$TD_j = D_j T'. \tag{7}$$

Clearly, (6) and (7) imply that

$$D_i^{-1} T D_i = T' = D_j^{-1} T D_j, \tag{8}$$

so

$$T D_i D_j^{-1} = D_i D_j^{-1} T. \tag{9}$$

Because all elements of $D_i D_j^{-1}$ are distinct by hypothesis, it follows that T is a diagonal matrix, and equivalence of X and Y is obtained. □

It should be noted that in the case where symmetry holds without equivalence, the very violation of the Harshman conditions permits a simplified expression for the nonequivalent parts of X and Y, from which equivalent solutions can also be found. Nevertheless, equivalence is not granted in that case.

An implication of Results 1 and 2 is that instead of examining equivalence of X and Y, we may focus on symmetry of the regression matrices $XD_1Y', \ldots, XD_mY'$. It should be obvious at once that symmetry is guaranteed in cases of perfect fit. It is also clear that symmetry is necessary and sufficient for equivalence if $r = 1$.

Fitting a Single Symmetric Matrix by CANDECOMP

For $m = 1$ and $Z_1 = S$ (symmetric), the function f reduces to

$$\mathrm{f}(X, Y) = \|S - XY'\|^2, \tag{10}$$

because $D_1 = D$ can be absorbed into X or Y, without loss of generality. Minimizing (10) by CANDECOMP can be interpreted as the $m = 1$ case of INDSCAL. Unrealistic as this may be, this case reveals how asymmetry may come about.

When X is optimal given Y, and Y is optimal given X, we have normal equations

$$Y' = (X'X)^{-1} X'S, \tag{11}$$

and

$$X' = (Y'Y)^{-1} Y'S. \tag{12}$$

Let P_x and P_y be defined as $P_x = X(X'X)^{-1}X'$ and $P_y = Y(Y'Y)^{-1}Y'$, respectively. Then (11) and (12) can be written equivalently as

$$P_x S = XY', \tag{13}$$

and

$$P_y S = YX', \tag{14}$$

respectively. The regression XY' is then

$$XY' = P_x S = S P_y = P_x S P_y. \tag{15}$$

Although XY' is often believed to be symmetric when (15) holds (e.g., Levin, 1988, p. 416), symmetry does not follow from (15). To show this, consider the case where

$$S^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

It is well-known that a best-fitting rank-one solution can be obtained by taking $X = Y = (\sqrt{.5}\ \sqrt{.5})'$, the first eigenvector of S^*, associated with the largest eigenvalue 1 of S^* (see Eckart & Young, 1936). However, asymmetric solutions also exist. For example, if $X' = (3 \quad 1)$ and $Y' = (.1 \quad .3)$, (15) is also satisfied, but XY' is now asymmetric; that is,

$$XY' = \begin{pmatrix} .3 & .9 \\ .1 & .3 \end{pmatrix}. \tag{17}$$

Both solutions yield a global minimum of 1 for f. The construction of the matrix S^* of (16) was inspired by a sufficient condition for symmetry of XY', to be derived shortly. First, however, it is convenient to obtain some preliminary results. Let $S = K\Lambda K'$ ($K'K = KK' = I$; Λ diagonal) be an eigendecomposition of S, and let X and Y be expressed in terms of K as

$$X = KU, \tag{18}$$

for some $n \times r$ matrix U. Also, let P_u be defined as

$$P_u = U(U'U)^{-1}U'. \tag{19}$$

Result 3. For $p = 1, 2, \ldots$, the matrix P_x commutes with S^p if and only if P_u commutes with Λ^p.

Proof. We have $P_xS^p = S^pP_x$ iff $X(X'X)^{-1}X'K\Lambda^pK' = K\Lambda^pK'X(X'X)^{-1}X'$ iff $KU(U'U)^{-1}U'\Lambda^pK' = K\Lambda^pU(U'U)^{-1}U'K'$ iff $P_u\Lambda^p = \Lambda^pP_u$. □

Result 4. If (15) holds, P_u commutes with Λ^2.

Proof. From (15), it follows that

$$P_xS^2 = SP_yS = S(SP_x) = S^2P_x, \tag{20}$$

and from Result 3 (for $p = 2$), $P_u\Lambda^2 = \Lambda^2P_u$. □

Result 5. If (15) holds, then XY' is symmetric if and only if P_u commutes with Λ.

Proof. Clearly, $XY' = YX'$ iff $SP_x = P_xS$ iff $\Lambda P_u = P_u\Lambda$, using Result 3 for $p = 1$. □

From now on, it is assumed that (15) holds. Result 4 then implies that $P_u\Lambda^2 = \Lambda^2P_u$, and the question is under what conditions do we have $P_u\Lambda = \Lambda P_u$. Because P_u commutes with Λ^2, P_u is a block-diagonal matrix with nonzero blocks along the diagonal, the orders of which correspond with the multiplicities of the diagonal elements of Λ^2. If the elements of Λ have the same multiplicities as those of Λ^2, then $P_u\Lambda = \Lambda P_u$ and symmetry holds (Result 5). It follows that asymmetry requires that, for a certain scalar λ, both λ and $-\lambda$ are eigenvalues of S. On the other hand, having such eigenvalues of opposite signs is not sufficient for asymmetry. In the remainder of the present section, necessary and sufficient conditions for asymmetry will be examined in detail.

For reasons of simplicity, let it be assumed that S has only one eigenvalue λ, with multiplicity s, such that $-\lambda$ is also an eigenvalue of S, with multiplicity t. Then K and Λ can be rearranged to the effect that

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \lambda I_s & 0 \\ 0 & 0 & -\lambda I_t \end{pmatrix} \equiv \left(\begin{array}{c|cc} \Lambda_1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\Lambda_*} \\ 0 & & \end{array}\right), \tag{21}$$

where Λ_1 is a diagonal matrix containing the $(n - s - t)$ eigenvalues of S that differ from λ and $-\lambda$. Clearly, Λ^2 then has the form

$$\Lambda^2 = \left(\begin{array}{c|ccc} \Lambda_1^2 & 0 & 0 & 0 \\ \hline 0 & \multicolumn{3}{c}{\lambda^2 I_{(s+t)}} \\ 0 & & & \end{array}\right), \tag{22}$$

and it follows from $P_u\Lambda^2 = \Lambda^2 P_u$ that P_u has the form

$$P_u = \left(\begin{array}{c|cc} P_1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{P_2} \\ 0 & & \end{array}\right), \tag{23}$$

where P_1 commutes with Λ_1^2 and also with Λ_1. Therefore, it depends entirely on P_2 whether or not P_u commutes with Λ.

If X and Y are replaced by XT and $Y(T^{-1})$, respectively, where T is an arbitrary nonsingular matrix, then neither XY', nor P_u, nor the validity of (15) are affected. Therefore, a T may be inserted that transforms U into a columnwise orthonormal matrix T_u, with $P_u = T_u T_u'$. Because of (23), T_u can further be rotated to an $n \times r$ matrix of the form

$$T_u = \left(\begin{array}{c|c} T_1 & 0 \\ \hline 0 & T_2 \\ 0 & \end{array}\right), \tag{24}$$

where T_2 has the order $(s + t) \times u$ for some $u \leq r$. Noting that $P_2 = T_2 T_2'$, it can be seen that P_u commutes with Λ if and only if T_2 can be rotated to a direct sum of an $s \times v$ matrix T_{2+} and a $t \times (u - v)$ matrix T_{2-} of the form

$$T_2 = \begin{matrix} & v & u-v \\ s & T_{2+} & 0 \\ t & 0 & T_{2-} \end{matrix} . \tag{25}$$

At this point it becomes clear how asymmetric solutions for XY' may come about. For instance, let $u = 1$. Then T_2 is a vector, and $T_2 T_2'$ commutes with Λ^* if and only if T_2 has zero elements throughout the first s or the last t elements. If there is at least one nonzero element both among the first s and among the last t elements of T_2, that is, if T_2 "truly mixes" eigenvectors associated with λ with eigenvectors associated with $-\lambda$, then asymmetry of XY' is guaranteed, and we have symmetry otherwise.

It is instructive to verify this result for the solution XY' in (17). The matrix S of

(16) has eigenvalues 1 and -1, and associated eigenvectors $\mathbf{k}'_1 = (\sqrt{.5}\ \sqrt{.5})$ and $\mathbf{k}'_2 = (\sqrt{.5}\ -\sqrt{.5})$, respectively. The matrix X was constructed as

$$X = \sqrt{2}(2\mathbf{k}_1 + \mathbf{k}_2) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \tag{26}$$

which is a true mixture of $\mathbf{k}_1$ and $\mathbf{k}_2$, thus producing asymmetry. Note that $P_u = P_2$ in this case.

It should be pointed out that constructing asymmetry is more involved when $u > 1$, because mixtures of eigenvectors can arise merely by rotating T_2. For the purposes of the present paper, however, it suffices to have a method of constructing asymmetric solutions in the $u = 1$ case. This will become clear in the next section when asymmetric INDSCAL solutions will be examined.

Asymmetric INDSCAL Solutions: The Indefinite Case

For $m = 1$, asymmetric solutions can be constructed if and only if $S_1 = S$ has one or more eigenvalues of opposite sign. This condition for asymmetry can readily be applied to INDSCAL with $m > 1$. Specifically, consider

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

and

$$S_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \tag{28}$$

both of which have eigenvalues of opposite sign. If we let

$$X = \begin{pmatrix} \sqrt{1/3} & \sqrt{.5} \\ -\sqrt{1/3} & 0 \\ \sqrt{1/3} & -\sqrt{.5} \end{pmatrix}, \tag{29}$$

and

$$Y = \begin{pmatrix} \sqrt{1/3} & \sqrt{.5} \\ \sqrt{1/3} & 0 \\ \sqrt{1/3} & -\sqrt{.5} \end{pmatrix}, \tag{30}$$

$D_1 = I_2$, and

$$D_2 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \tag{31}$$

then X and Y are nonequivalent and XD_1Y' and XD_2Y' are asymmetric. This solution yields a global minimum for the CANDECOMP function, as can be verified as follows. The regression matrices are

$$XD_1Y' = 1/6 \begin{pmatrix} 5 & 2 & -1 \\ -2 & -2 & -2 \\ -1 & 2 & 5 \end{pmatrix}, \tag{32}$$

and

$$XD_2Y' = 1/6 \begin{pmatrix} -2 & 4 & 10 \\ -4 & -4 & -4 \\ 10 & 4 & -2 \end{pmatrix}, \tag{33}$$

which implies that the residual sum of squares (1) is 5. From Eckart and Young (1936) it is well-known that the best Rank 2 approximation of a symmetric matrix yields a residual sum of squares equal to the smallest squared eigenvalue. These values are 1 and 4, for S_1 and S_2, respectively. It follows that 5 is a lower bound to the CANDECOMP loss function (1) for this S_1 and S_2, and that we have asymmetric solutions for INDSCAL at a global minimum of the CANDECOMP function.

The existence of asymmetric solutions does not exclude the existence of symmetric solutions, nor does it imply that the CANDECOMP algorithm will converge to asymmetric solutions if started anywhere but at the X and Y of (29) and (30). The main purpose of considering asymmetric examples is to narrow down the variety of conditions, under which symmetry proofs are feasible.

It may be conjectured that asymmetry cannot occur at stationary points of the CANDECOMP function if the matrices $S_1, \ldots, S_m$ are *Gramian*. Surprisingly, however, even this conjecture is false, as will be shown in the next section.

Asymmetric INDSCAL Solutions: The Gramian Case

Let a stationary point of the CANDECOMP function be defined as a set of values for X, Y, and $D_1, \ldots, D_m$ satisfying the normal equations

$$Y = \sum_{i=1}^{m} S_i X D_i \left(\sum_{j=1}^{m} D_j X' X D_j \right)^{-1}, \tag{34}$$

$$X = \sum_{i=1}^{m} S_i Y D_i \left(\sum_{j=1}^{m} D_j Y' Y D_j \right)^{-1}, \tag{35}$$

and, for $i = 1, \ldots, m$,

$$\text{Diagvec}\,(D_i) = (X'X * Y'Y)^{-1}\ \text{Diagvec}\,(X'S_iY), \tag{36}$$

where $*$ denotes the elementwise (Hadamard) product of matrices, and Diagvec $(\cdot)$ denotes the vector containing the diagonal elements of the matrix between parentheses. Clearly, the CANDECOMP process is terminated if and only if these equations are satisfied jointly, within limits of computational accuracy.

The following example represents a stationary point of the CANDECOMP function for Gramian matrices: let

$$S_1 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{37}$$

and

$$S_2 = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{37}$$

and consider the $r = 1$ solution

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \; Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{39}$$

with $D_1 = [1]$ and $D_2 = [-1]$. It is evident that the solution lacks equivalence and symmetry. It can be verified that neither S_1 nor S_2 have eigenvalues of opposite sign, but $(S_1 - S_2)$ does.

It is important to see that the solution (39) is not globally optimal. That is, the value for f obtained is 39 whereas it can be shown that f $\geq$ 21, with equality and hence the global minimum 21 if we take

$$X = Y = \begin{pmatrix} \sqrt{.5} \\ \sqrt{.5} \\ 0 \end{pmatrix}, \tag{40}$$

with $D_1 = [4]$ and $D_2 = [2]$. Disregarding joint reflection of X and Y, there is only one other globally optimal solution, namely

$$X = Y = \begin{pmatrix} \sqrt{.5} \\ -\sqrt{.5} \\ 0 \end{pmatrix}, \tag{41}$$

with $D_1 = [2]$ and $D_2 = [4]$. It follows that asymmetry in the present example can only occur at *local* optima of the CANDECOMP function. In fact, this property is of a more general nature, as will be explained below. First, however, it is instructive to examine the nature of the nonoptimality of solution (39). It is easy to verify that the X and Y of (39) are globally optimal given D_1 and D_2. That is, no other pair X^*, Y^* can yield a lower residual sum of squares for the D_1 and D_2 given. This shows that any attempt to prove symmetry, for Gramian matrices $S_1, \ldots, S_m$, using merely the joint optimality of X and Y, conditional on D_1 and D_2, is doomed to fail. It seems that global optimality of $D_1, \ldots, D_m$ must be included in any proof of symmetry for Gramian matrices.

Alternatively, one might expect the matrices $D_1, \ldots, D_m$ to be nonnegative at the global minimum for Gramian matrices, and hence, be tempted to prove symmetry for optimal X and Y, given nonnegative matrices $D_1, \ldots, D_m$. However, it is not difficult to construct counterexamples showing that globally optimal INDSCAL solutions for Gramian matrices can have negative elements in $D_1, \ldots, D_m$.

A proof for symmetry at the global minimum of the CANDECOMP function for Gramian matrices has not been found, nor have any counterexamples occurred. However, for Gramian matrices symmetry can be shown to hold at solutions that minimize (1) subject to the constraint that X and Y are orthonormal columnwise. This will be taken up in the next section.

INDSCAL with Orthonormality Constraints

Kroonenberg (1983, p. 118) considered minimizing the INDSCAL loss function (2) subject to the constraint $X'X = I_r$. An algorithm that accomplishes this for Gramian matrices has been given by ten Berge, Knol, and Kiers (1988). The algorithm is based on the observation that the CANDECOMP function, constrained by $X'X = Y'Y = I$, can be simplified to

$$f(X, Y, D_1, \ldots, D_m) = \sum_{i=1}^{m} (\text{tr } Z_i'Z_i - 2 \text{ tr } Z_i'XD_iY' + \text{tr } D_i^2). \tag{42}$$

It is straightforward that the optimal $D_1, \ldots, D_m$ can now be expressed in terms of X and X as

$$D_i = \text{Diag } (X'Z_iY) = \text{Diag } (X'S_iY), \tag{43}$$

$i = 1, \ldots, m$, in the constrained INDSCAL case. Accordingly, the problem that remains is to maximize

$$h(X, Y) = \sum_{i=1}^{m} \text{tr } (\text{Diag } X'S_iY)^2, \tag{44}$$

subject to $X'X = Y'Y = I_r$. This sets the stage for the following equivalence result:

Result 6. At every stationary point of h(X, Y) we have $X = Y$ if $S_1, \ldots, S_m$ are Gramian and at least one of these matrices, S_j say, is nonsingular.

Proof. For $i = 1, \ldots, m$, we have

$$\|\text{Diag } (X'S_iX) - \text{Diag } (Y'S_iY)\|^2 \geq 0; \tag{45}$$

hence, summing over i yields

$$h(X, X) + h(Y, Y) \geq 2 \sum_{i=1}^{m} \text{tr } \{(\text{Diag } X'S_iX)(\text{Diag } Y'S_iY)\}. \tag{46}$$

From the Schwartz inequality applied to $(S_i^{1/2}\mathbf{x}_l)'(S_i^{1/2}\mathbf{y}_l)$, we have

$$h(X, Y) = \sum_{i=1}^{m} \sum_{l=1}^{r} (\mathbf{x}_l'S_i\mathbf{y}_l)^2 \leq \sum_{i=1}^{m} \sum_{l=1}^{r} (\mathbf{x}_l'S_i\mathbf{x}_l)(\mathbf{y}_l'S_i\mathbf{y}_l) \tag{47}$$

$$= \text{tr} \sum_{i=1}^{m} \{(\text{Diag } X'S_iX)(\text{Diag } Y'S_iY)\}.$$

Combining (46) and (47) shows that

$$h(X, X) + h(Y, Y) \geq 2 \text{ h}(X, Y). \tag{48}$$

At stationary points of h we have X optimal given Y, and Y optimal given X, implying that

$$2 \text{ h}(X, Y) = h(X, Y) + h(X, Y) \geq h(X, X) + h(Y, Y), \tag{49}$$

and it follows that (48) holds as an equality. This implies that, for $l = 1, \ldots, r$, $S_j^{1/2}\mathbf{x}_l = \lambda_l S_j^{1/2}\mathbf{y}_l$ ($\lambda_l > 0$), and the inference $\mathbf{x}_l = \mathbf{y}_l$ ($l = 1, \ldots, r$) is immediate. □

Result 6 has no direct bearing on CANDECOMP except for the case $r = 1$. The global minimum of CANDECOMP, applied to Gramian matrices, coincides with the global maximum of h(X, Y) if $r = 1$, and equivalence is guaranteed at this global maximum. This explains why (39) and similar counterexamples could only be constructed for nonoptimal solutions. Unfortunately, for $r > 1$ the orthonormality constraint is active and Result 6 is accordingly of no avail in proving symmetry for unconstrained CANDECOMP at the global minimum.

Discussion

The question whether or not equivalence for X and Y holds upon convergence of CANDECOMP applied to Gramian matrices remains unsettled, except for $r = 1$. However, we have narrowed down the range of possibilities by showing that symmetry proofs for Gramian $S_1, \ldots, S_m$, if possible at all, must rely on more than the conditional optimality of X and Y given $D_1, \ldots, D_m$ alone. Also, it cannot be taken for granted that $D_1, \ldots, D_m$ will be nonnegative at global minima of the CANDECOMP function. From a practical point of view one might be tempted to avoid asymmetry by setting X equal to Y in each computational cycle of CANDECOMP. Carroll and Chang (1970, p. 288) have explicitly considered and rejected such an approach, for good reasons: as the parameters approach conditional optimality, setting X equal to Y may *increase* the residual sum of squares, thus disturbing the monotonicity of CANDECOMP. Therefore, setting X equal to Y in each cycle hardly contributes to the mathematical treatment of symmetry in INDSCAL.

References

Carroll, J. D., & Chang, J. J. (1970) Analysis of individual differences in multidimensional scaling via an n-way generalization of "Eckart-Young" decomposition. *Psychometrika, 35*, 283–319.

Carroll, J. D., & Pruzansky, S. (1984) The CANDECOMP-CANDELINC family of models and methods for multidimensional data analysis. In H. G. Law, C. W. Snyder, J. A. Hattie, & R. P. McDonald (Eds.), *Research methods for multimode data analysis* (pp. 372–402). New York: Praeger.

Eckart, C., & Young, G. (1936) The approximation of one matrix by another of lower rank. *Psychometrika, 1*, 211–218.

Harshman, R. A. (1970) Foundations of the PARAFAC procedure: models and conditions for an "explanatory" multi-mode factor analysis. *UCLA Working Papers in Phonetics, 16*, 1–84.

Harshman, R. A. (1972) Determination and proof of minimum uniqueness conditions for PARAFAC1. *UCLA Working Papers in Phonetics, 22*, 111–117.

Kroonenberg, P. M. (1983). *Three mode principal component analysis: Theory and applications*. Leiden: DSWO Press.

Levin, J. (1988). Note on convergence of MINRES. *Multivariate Behavioral Research, 23*, 413–417.

ten Berge, J. M. F., Kiers, H. A. L., & de Leeuw, J. (1988) Explicit CANDECOMP/PARAFAC solutions for a contrived 2 × 2 × 2 array of rank three. *Psychometrika, 53*, 579–584.

ten Berge, J. M. F., Knol, D. L., & Kiers, H. A. L. (1988). A treatment of the orthomax rotation family in terms of diagonalization, and a re-examination of a singular value approach to varimax rotation. *Computational Statistics Quarterly, 3*, 207–217.

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