JOURNAL OF CHEMOMETRICS
J. Chemometrics 2004; **18**: 12–16
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/cem.839

Partial uniqueness in CANDECOMP/PARAFAC

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Received 4 July 2003; Revised 7 November 2003; Accepted 5 January 2004

A key property of CANDECOMP/PARAFAC is the essential uniqueness it displays under certain conditions. It has been known for a long time that, when these conditions are not met, partial uniqueness may remain. Whereas considerable progress has been made in the study of conditions for uniqueness, the study of partial uniqueness has lagged behind. The only well known cases are those of overfactoring, when more components are extracted than are required for perfect fit, and those cases where the data do not have enough system variation, resulting in proportional components for one or more modes. The present paper deals with partial uniqueness in cases where the smallest number of components is extracted that yield perfect fit. For the case of $K \times K \times 2$ arrays of rank K, randomly sampled from a continuous distribution, it is shown that partial uniqueness, with some components unique and others differing between solutions, arises with probability zero. Also, a closed-form CANDECOMP/PARAFAC solution is derived for $5 \times 3 \times 3$ arrays when these happen to have rank 5. In such cases, any two different solutions share four of the five components. This phenomenon will be traced back to a sixth degree polynomial having six real roots, any five of which can be picked to construct a solution. Copyright © 2004 John Wiley & Sons, Ltd.

KEYWORDS: CANDECOMP/PARAFAC; uniqueness; typical rank

1. INTRODUCTION

Carroll and Chang [1] and Harshman [2] independently proposed CANDECOMP and PARAFAC respectively. Let $\underline{\mathbf{X}}$ be a three-way array containing K slices $\mathbf{X}_k$, $k = 1, \ldots, K$, of order $I \times J$. For a fixed number of components r a least squares fit of CANDECOMP/PARAFAC (CP) yields component matrices $\mathbf{A}$ $(I \times r)$, $\mathbf{B}$ $(J \times r)$ and $\mathbf{C}$ $(K \times r)$ minimizing $\Sigma\text{tr}(\mathbf{E}_k'\mathbf{E}_k)$ in the decomposition

$$\mathbf{X}_k = \mathbf{A}\mathbf{C}_k\mathbf{B}' + \mathbf{E}_k \tag{1}$$

$k = 1, \ldots, K$, where $\mathbf{C}_k$ is the diagonal matrix containing the elements of row k of $\mathbf{C}$.

CP has gained popularity in chemistry, e.g. for performing second-order calibration or resolving mixtures in fluorescence spectroscopy, owing to its property of (essential) *uniqueness*. That is, under mild conditions to be discussed below, CP solutions are identified up to rescaling and joint permutation of the columns of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. When the CP solution fails to meet the conditions for uniqueness, we often have full non-uniqueness. However, cases of partial uniqueness, where different solutions have some components in common, have also been discussed [2]. Compared to the advances that have been made in the study of uniqueness, little is known about partial uniqueness. In the present paper we examine two cases of partial uniqueness. In the first case, non-uniqueness will be shown to arise with probability zero when the array values are randomly sampled from a continuous distribution. In the second case, any two solutions have most of their components in common, yet no single component is guaranteed to be common to every solution. Although such a pattern has been observed before in situations of overfactoring [2], it now occurs, surprisingly, when the correct number of components is extracted.

The organization of this paper is as follows. First we briefly review the most relevant uniqueness results. Next we consider some recent results about the number of components required for a perfect fit in CP under random sampling of the array elements from a continuous distribution. This number is the so-called 'typical rank' of three-way arrays. It will be shown that cases of perfect fit usually fail to display uniqueness. Finally we discuss two cases of partial uniqueness when the correct number of components is extracted. In passing, the typical rank of $5 \times 3 \times 3$ arrays will be determined.

2. A REVIEW OF UNIQUENESS IN CP

The first uniqueness results of CP date back to Jennrich (quoted in Reference [2]) and Harshman [3]. The most general sufficient condition for uniqueness is due to Kruskal [4]. Kruskal's condition relies on a particular concept of matrix rank that he introduced, which has been named k-rank (Kruskal rank) after him by Harshman and Lundy [5]. Specifically, a matrix $\mathbf{A}$ has k-rank k_A when every k_A columns of $\mathbf{A}$ are linearly independent and at least one set of $k_\text{A} + 1$ columns are not. Kruskal [4] proved that the condition

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$k_A + k_B + k_C \geqslant 2r + 2$ is sufficient for essential uniqueness in CP. More than two decades later the study of uniqueness has been revived. Sidiropoulos and Bro [6] have given a short-cut proof for Kruskal's sufficient condition and generalized it to n-way arrays ($n > 3$). Ten Berge and Sidiropoulos [7] have shown that Kruskal's *sufficient* condition is also *necessary* for $r = 2$ or 3, but not for $r > 3$. In practice, the condition is almost invariably met, because the number of components is usually small enough. It may be noted that the condition cannot be met when $r = 1$. However, uniqueness for that case has already been proven by Harshman [3].

Harshman [2] (p. 39) suggested using the occurrence of uniqueness as a diagnostic for the correct number of factors. For 'adequate' data sets, having enough systematic trilinear variation to entail CP solutions with full linear independence of their columns in two of the three modes and non-proportional columns in the third, he found that CP uniqueness was inherent to having the 'correct' number of components or less and that non-uniqueness in adequate data sets arose only from overfactoring. The data sets studied by Harshman were generated 'synthetically' by choosing random matrices **A**, **B** and **C** of a given order and deriving the data $\mathbf{X}_1, \ldots, \mathbf{X}_K$ from the CP model (1) with zero error matrices $\mathbf{E}_k$, $k = 1, \ldots, K$. It is tempting to infer from Harshman's results that a similar relation between rank and uniqueness (non-uniqueness, for adequate data sets, is due to overfactoring) will hold when the three-way array is not generated synthetically through **A**, **B** and **C** fed into the CP model, but instead $\mathbf{X}_1, \ldots, \mathbf{X}_K$ are randomly sampled at once and CP is applied with the smallest dimensionality that allows perfect fit. That temptation, as will be shown below, should be resisted. The rank that will be needed for perfect fit when the three-way array itself is randomly generated is the so-called 'typical rank' of the array. It is the topic of the next section.

3. SOME RECENT RESULTS ON TYPICAL RANK

It is well known that the maximal rank a *matrix* of a given order may have is also the typical rank, i.e. the rank that will be observed when the matrix is randomly filled with elements from a continuous distribution. For instance, when a 5×3 matrix is constructed randomly, its rank is 3 with probability one. For three-way arrays the maximal rank is often higher than the typical rank. The first examples were given by J. B. Kruskal (unpublished manuscript, 1983). They have recently been generalized by ten Berge and Kiers [8] and ten Berge [9]. Typical rank is the rank an array has with positive probability when its elements are drawn randomly from a continuous distribution. In most cases this appears to be a single number, but there are also array formats having two different rank values with positive probability. For instance, a $2 \times 4 \times 4$ array has rank 4 with probability P, $0 < P < 1$, and it has rank 5 with probability $1-P$, whereas the maximal rank is 6. Arrays of format $2 \times 4 \times 4$ with rank 0, 1, 2, 3 and 6 can be constructed but will never be encountered in practice. For all practical purposes, typical rank is the smallest number of components that allow perfect fit in CP. That is, when a three-way array is filled with real-life data from a random sample of subject/objects on a number of measures at a number of occasions, the array rank will be the typical rank. Admittedly, real life samples are drawn from discrete distributions, thus invalidating the assumption of continuity. However, the chances of finding atypical rank values in such cases are too remote to be of any practical concern. An overview of known typical ranks for arrays containing two or three slices is given in Table I below, repeated from Reference [9].

At this point, we may verify, for each of the arrays for which the typical rank is known, whether or not uniqueness is possible in cases of perfect fit. Because the k-rank of a matrix cannot exceed its number of rows, having $k_A + k_B + k_C \geqslant 2r + 2$ (Kruskal's condition) is impossible whenever $I + J + K < 2r + 2$. In addition, the k-ranks of **A**, **B** and **C** cannot exceed r. It follows that the only cases of Table I that might involve uniqueness by virtue of Kruskal's condition are the $2 \times 2 \times 2$ array when it has rank 2, the $3 \times 3 \times 2$ array when it has rank 3, and the $4 \times 4 \times 2$ array when it has rank 4. In fact, these are indeed the only cases (the question marks and omitted cases in Table I included) known to display uniqueness in numerical experiments. It can be concluded that non-uniqueness is quite common in situations where the correct number of factors is extracted. Still, non-unique cases of perfect fit may display *partial* uniqueness. Two of such cases will be examined in the remaining sections.

4. PARTIAL UNIQUENESS IN A CASE OF PERFECT FIT: THE $K \times K \times 2$ ARRAYS OF RANK K

Harshman [3] has proven uniqueness for CP solutions when two component matrices have full column rank and the third has no proportional columns. This implies that $K \times K \times 2$ arrays of rank K will have a unique CP solution when **A** and **B** are non-singular and the $2 \times K$ matrix **C** has no proportional columns. When **C** does have proportional columns, partial uniqueness will remain for the components associated with the other (non-proportional) columns of **C**. It will now be shown that having proportional columns in **C** is an event of probability zero when the array is sampled randomly.

Let the slices be $\mathbf{X}_1$ and $\mathbf{X}_2$. Suppose that a rank-K solution exists. Then we can solve $\mathbf{X}_1 = \mathbf{A}\mathbf{C}_1\mathbf{B}'$ and $\mathbf{X}_2 = \mathbf{A}\mathbf{C}_2\mathbf{B}'$, with **A** and **B** square and non-singular. Hence $\mathbf{X}_1\mathbf{X}_2^{-1} = \mathbf{A}\mathbf{C}_1\mathbf{C}_2^{-1}\mathbf{A}^{-1}$, which is an eigenvalue decomposition. Because a solution in K components is possible by assumption, all eigenvalues of $\mathbf{X}_1\mathbf{X}_2^{-1}$ are real. When these eigenvalues

Table I. Typical rank results for some arrays with $K = 2$ and $K = 3$

$K=2$				$K=3$			
	$J=2$	$J=3$	$J=4$		$J=3$	$J=4$	$J=5$
$I=2$	{2,3}	3	4	$I=5$	?	?	?
$I=3$	3	{3,4}	4	$I=6$	6	?	?
$I=4$	4	4	{4,5}	$I=7$	7	?	?
$I=5$	4	5	5	$I=8$	8	{8,9}	?
$I=6$	4	6	6	$I=9$	9	9	?
$I=7$	4	6	7	$I=10$	9	10	10
$I=8$	4	6	8	$I=11$	9	11	11
$I=9$	4	6	8	$I=12$	9	12	12

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are distinct, all eigenvectors are essentially unique and **A** and **B** are determined uniquely up to permutation and scale. However, for any pair of equal eigenvalues the associated two eigenvectors are determined up to a non-singular transformation. Here we have a case of partial uniqueness. Every solution will contain the same components that depend on the distinct eigenvalues, but the other components are indeterminate.

The partial uniqueness encountered here arises when a matrix that depends only on the data in $\mathbf{X}_1$ and $\mathbf{X}_2$ has two (or more) equal eigenvalues. It is well known that a square matrix has two or more equal eigenvalues with probability zero when the array is drawn randomly from a continuous distribution. Therefore we may dismiss this case as one of no practical concern. When a randomly generated $K \times K \times 2$ array has rank K, and CP extracts K components, Harshman's sufficient condition for uniqueness will be satisfied almost surely.

Harshman [2] (see also References [5,10,11]) discussed another type of partial uniqueness in a situation of overfactoring. He reported cases where each solution shares some components with the solution from which data were generated, yet there is no part of the solution common to every solution. This type of non-uniqueness also appears in cases with no overfactoring. The next example demonstrates this.

5. PARTIAL UNIQUENESS IN A CASE OF PERFECT FIT: THE $5 \times 3 \times 3$ ARRAY OF RANK 5

Numerical tests with random $5 \times 3 \times 3$ arrays suggest that they have rank 6 in most cases, yet sometimes they have rank 5. Also, whenever the rank is 5, different solutions invariably appear to have four out of five components in common (there appears to be no uniqueness at all when the rank is 6). In this section it will first be proven that the $5 \times 3 \times 3$ array has either rank 5 or rank 6 with positive probability. A closed-form solution will be derived for the case when the array has rank 5. From this solution the nature of the partial uniqueness will be fully clarified.

Let the $5 \times 3 \times 3$ array have slices **X**, **Y** and **Z** of order 5×3. By the method of ten Berge and Kiers [8], the array can be transformed to have

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The transformation is rank-preserving and merely serves to simplify the problem. We examine under which conditions a rank-5 CP solution is possible. That is, we desire a non-singular 5×5 matrix **A**, a 3×5 matrix **B** and three non-singular diagonal 5×5 matrices $\mathbf{I}_5$, **C** and **D** such that

$$\mathbf{X} = \mathbf{A}\mathbf{I}_5\mathbf{B}', \qquad \mathbf{Y} = \mathbf{A}\mathbf{C}\mathbf{B}', \qquad \mathbf{Z} = \mathbf{A}\mathbf{D}\mathbf{B}' \tag{3}$$

Note that the first of the three diagonal matrices has been set to $\mathbf{I}_5$ by absorbing its elements into **A**. Suppose the construction (3) is possible. Then

$$\mathbf{A}^{-1}\mathbf{X} = \mathbf{B}', \qquad \mathbf{A}^{-1}\mathbf{Y} = \mathbf{C}\mathbf{B}', \qquad \mathbf{A}^{-1}\mathbf{Z} = \mathbf{D}\mathbf{B}' \tag{4}$$

whence

$$\mathbf{C}\mathbf{A}^{-1}\mathbf{X} - \mathbf{A}^{-1}\mathbf{Y} = \mathbf{D}\mathbf{A}^{-1}\mathbf{X} - \mathbf{A}^{-1}\mathbf{Z} = \mathbf{O} \tag{5}$$

At this point, note that **B** has been removed from the equations. It remains to find **A**, **C** and **D**. Let $\mathbf{a}_j'$ be row j of $\mathbf{A}^{-1}$. Then we need to determine five linearly independent solutions to the vector equation $\mathbf{a}_j'(c_j\mathbf{X} - \mathbf{Y}) = \mathbf{a}_j'(d_j\mathbf{X} - \mathbf{Z}) = \mathbf{0}'$, where c_j and d_j are the jth diagonal elements of **C** and **D** respectively. Set the first element of $\mathbf{a}_j$ to 1. Owing to (2), the first part of the vector equation, i.e. $\mathbf{a}_j'(c_j\mathbf{X} - \mathbf{Y}) = \mathbf{0}'$, is equivalent to

$$\mathbf{a}_j' = [1 \; e_j \; c_j \; c_je_j \; c_j^2] \tag{6}$$

for some scalar e_j. It remains to satisfy $\mathbf{a}_j'(d_j\mathbf{X} - \mathbf{Z}) = \mathbf{0}'$. Let **Z** have columns **f**, **g** and **h**. Then

$$\mathbf{Z} - d_j\mathbf{X} = \begin{bmatrix} f_1 - d_j & g_1 & h_1 \\ f_2 & g_2 - d_j & h_2 \\ f_3 & g_3 & h_3 - d_j \\ f_4 & g_4 & h_4 \\ f_5 & g_5 & h_5 \end{bmatrix} \tag{7}$$

In order to get $\mathbf{a}_j$ orthogonal to the columns of $\mathbf{Z} - d_j\mathbf{X}$, we have to get

$$\begin{bmatrix} 1 \\ e_j \end{bmatrix}$$

orthogonal to columns of a 2×3 matrix $\mathbf{W}_j$ having, as first row, row 1 plus c_j times row 3 plus c_j^2 times row 5 of $\mathbf{Z} - d_j\mathbf{X}$, the second row consisting of row 2 plus c_j times row 4 of that matrix. Then

$$\begin{bmatrix} 1 \\ e_j \end{bmatrix}$$

must be determined to be orthogonal to the columns of

$$\mathbf{W}_j = \begin{bmatrix} c_j^2f_5 + c_jf_3 + f_1 - d_j & c_j^2g_5 + c_jg_3 + g_1 & c_j^2h_5 + c_j(h_3 - d_j) + h_1 \\ c_jf_4 + f_2 & c_jg_4 + g_2 - d_j & c_jh_4 + h_2 \end{bmatrix} \tag{8}$$

Clearly, to get such an e_j, we need $\mathbf{W}_j$ to be of rank 1. We shall demand that the submatrix of columns 1 and 2 and that of columns 1 and 3 have determinant zero. However, this is not enough. It is possible to get both determinants zero without having $\mathbf{W}_j$ of rank 1: just pick c_j and d_j such that column 1 of $\mathbf{W}_j$ vanishes, and the determinants vanish. Accordingly, we shall have to remove that flawed solution with $c_j = -f_2/f_4$ and $d_j = c_j^2f_5 + c_jf_3 + f_1$ afterwards. Because column 1 is proportional to column 3, we can solve explicitly for d_j from

$$\begin{aligned}(c_j^2f_5 + c_jf_3 + f_1)(c_jh_4 + h_2) - (c_jf_4 + f_2)(c_j^2h_5 + c_jh_3 + h_1) \\ -d_j(c_jh_4 + h_2) + d_j(c_j^2f_4 + c_jf_2) = 0\end{aligned} \tag{9}$$

With (9), d_j can be removed from $\mathbf{W}_j$. The last step is to use that columns 1 and 2 of $\mathbf{W}_j$ are proportional. This entails a seventh-degree polynomial in c_j. The derivation of the coefficients $z_7, \ldots, z_0$ is given in the Appendix. The seven roots can be evaluated explicitly by the Matlab command roots($[z_7, \ldots, z_0]$). Because the flawed root $-f_2/f_4$ has to be discarded, we are essentially looking at a sixth-degree polynomial equation. When it has six real roots, a rank-5 solution is immediate. For example, suppose that, after simplifying **X**

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and $\mathbf{Y}$ as in (2), we have

$$\mathbf{Z} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad (10)$$

Then the Appendix implies that we take roots([1 − 3 − 67 10 335 288 21 − 5]), which yields the seven real roots 9.5098, −6.4927, 2.6391, −1.5467, 0.0964, −0.2058 and −1. The last root is-f_2/f_4 and has to be removed. We may pick any five of the six remaining roots to fill $\mathbf{C}$, and the elements of $\mathbf{D}$ are implied by (9). The five values of e_j follow immediately from orthogonality of

$$\begin{bmatrix} 1 \\ e_j \end{bmatrix}$$

to an arbitrary column of $\mathbf{W}_j$. Finally, the rows of $\mathbf{A}^{-1}$ are obtained from (6). After constructing five different rows for $\mathbf{A}^{-1}$, $\mathbf{B}$ follows from $\mathbf{B}' = \mathbf{A}^{-1}\mathbf{X}$; see Equation (4).

In general, the polynomial has all roots real-valued with small but positive probability. Empirically, this can be seen by slightly changing the elements of the $\mathbf{Z}$ in (10) and noting that all roots remain real. When the polynomial has less than seven real roots, it has at least one pair of complex roots and hence it has less than six real roots, one of which is flawed. Therefore a rank-5 solution does not exist in this case. For such cases, rank 6 is enough, because a $6 \times 3 \times 3$ array has typical rank 6 (see Table I), hence a $5 \times 3 \times 3$ array has at most typical rank 6. Rank less than 5 has probability zero, because the nine columns of the array span a five-dimensional space almost surely. It has thus been proven that the $5 \times 3 \times 3$ array has typical rank {5, 6}, which removes a question mark from Table I. It remains to be seen what partial uniqueness is implied when the array has rank 5.

Each of the six viable roots gives a value of c_j entailing values for d_j and e_j. Because only one root can be discarded, there are exactly six different solutions for the rows of $\mathbf{A}^{-1}$ and the corresponding diagonal elements of $\mathbf{C}$ and $\mathbf{D}$. This already implies that, unless they coincide, any two solutions share precisely four rows of $\mathbf{A}^{-1}$ and four diagonal elements in $\mathbf{C}$ and in $\mathbf{D}$. In addition, from $\mathbf{AB}' = \mathbf{X}$ (see Equation (2)), we have that the columns of $\mathbf{B}'$ are the first three columns of $\mathbf{A}^{-1}$. From (6) it can be seen that the columns of $\mathbf{B}$ have elements 1, e_j and c_j. It follows that every pair of different solutions has also four columns of $\mathbf{B}$ in common. Finally, consider $\mathbf{A}$ itself. Every column of $\mathbf{A}$ is orthogonal to four rows of $\mathbf{A}^{-1}$. Because every pair of different CP solutions share four rows of $\mathbf{A}^{-1}$, their solutions for $\mathbf{A}$ must share precisely one column. The partial uniqueness phenomenon for $5 \times 3 \times 3$ arrays is thus fully understood.

The type of partial uniqueness found here resembles what Harshman [2] found in cases of overfactoring with synthetic data. There is one difference. Whereas a single component recovered in two different solutions described in Reference [2] consists of one column of $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{A}$ respectively, in our case it consists of one column of $\mathbf{C}$, one column of $\mathbf{B}$ and one row of *the inverse* of $\mathbf{A}$ respectively.

6. DISCUSSION

Two cases of partial uniqueness in a case of perfect fit have been discussed. In the first case, when a $K \times K \times 2$ array, sampled randomly from a continuous distribution, happens to have rank K, partial uniqueness would imply that only a subset of components is unique and that there is an infinite set of solutions for the non-unique components. It has been shown that this specific case of partial uniqueness will not arise in practice, because Harshman's [3] sufficient condition for CP uniqueness will be satisfied almost surely. No assumptions were made about the systematic and error components in the three-way design, since all the mathematical results derived herein hold without such assumptions.

The second type of partial uniqueness discussed arises when the $5 \times 3 \times 3$ array happens to have rank 5. In that case there are exactly six solutions, and each pair must share at least four components, but no component is common to every solution. In passing, we have removed a question mark from Table I by showing that the typical rank of $5 \times 3 \times 3$ arrays is {5, 6}. Whereas previous simulation studies have revealed this type of partial uniqueness in cases of overfactoring, it now arises when the correct number of factors extracted. This implies that using uniqueness as a potential criterion for determining the correct number of factors is not as straightforward as one might wish.

Acknowledgements
Without holding them responsible for any of the views adopted here, critical comments by Richard Harshman, Nikos Sidiropoulos and Alwin Stegeman are gratefully acknowledged.

APPENDIX. THE COEFFICIENTS OF THE SEVENTH-DEGREE POLYNOMIAL

Dropping the subscripts from c_j and d_j, we have from (9) that

$$d = \frac{(c^2 f_5 + c f_3 + f_1)(c h_4 + h_2) - (c^2 h_5 + c h_3 + h_1)(c f_4 + f_2)}{-c^2 f_4 + c h_4 - c f_2 + h_2} =$$
$$\frac{c^3(f_5 h_4 - f_4 h_5) + c^2(f_5 h_2 - f_2 h_5 + f_3 h_4 - f_4 h_3) + c(f_1 h_4 - f_4 h_1 + f_3 h_2 - f_2 h_3) + f_1 h_2 - f_2 h_1}{-c^2 f_4 + c(h_4 - f_2) + h_2}$$

which can be written as A/B. Write $A = x_3 c^3 + x_2 c^2 + x_1 c + x_0$ and $B = y_2 c^2 + y_1 c + y_0$ for the known scalars x_3, x_2, x_1, x_0, y_2, y_1 and y_0. Then it remains to get c such that columns 1 and 2 of $\mathbf{W}_j$ are proportional. This means that

$$d^2 - d(c g_4 + g_2 + c^2 f_5 + c f_3 + f_1) + (c^2 f_5 + c f_3 + f_1)(c g_4 + g_2) - (c^2 g_5 + c g_3 + g_1)(c f_4 + f_2) = 0$$

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This can be written as

$$d^2 + d(v_2c^2 + v_1c + v_0) + (w_3c^3 + w_2c^2 + w_1c + w_0) = 0$$

for known scalars $v_2, v_1, v_0, w_3, w_2, w_1$ and w_0. Thus we need to solve

$$A^2 + AB(v_2c^2 + v_1c + v_0) + B^2(w_3c^3 + w_2c^2 + w_1c + w_0) = 0$$

where

$$A^2 = x_3^2c^6 + 2x_3x_2c^5 + (x_2^2 + 2x_3x_1)c^4 + (2x_3x_0 + 2x_2x_1)c^3 + (x_1^2 + 2x_2x_0)c^2 + 2x_1x_0c + x_0^2$$

$$AB = x_3y_2c^5 + (x_2y_2 + x_3y_1)c^4 + (x_3y_0 + x_2y_1 + x_1y_2)c^3 + (x_2y_0 + x_1y_1 + x_0y_2)c^2 + (x_1y_0 + x_0y_1)c + x_0y_0$$

$$B^2 = y_2^2c^4 + 2y_2y_1c^3 + (y_1^2 + 2y_2y_0)c^2 + 2y_1y_0c + y_0^2$$

This leads to a seventh degree polynomial in c. The coefficients z_j of $c^j, j = 7, 6, \ldots, 1, 0$, are

$$z_7 = x_3y_2v_2 + w_3y_2^2$$

$$z_6 = x_3^2 + x_3y_2v_1 + (x_2y_2 + x_3y_1)v_2 + y_2^2w_2 + 2y_1y_2w_3$$

$$z_5 = 2x_3x_2 + x_3y_2v_0 + (x_2y_2 + x_3y_1)v_1 + (x_3y_0 + x_2y_1 + x_1y_2)v_2 + y_2^2w_1 + 2y_2y_1w_2 + (y_1^2 + 2y_2y_0)w_3$$

$$z_4 = (x_2^2 + 2x_3x_1) + (x_2y_2 + x_3y_1)v_0 + (x_3y_0 + x_2y_1 + x_1y_2)v_1 + (x_2y_0 + x_1y_1 + x_0y_2)v_2 + y_2^2w_0 + 2y_2y_1w_1 + (y_1^2 + 2y_2y_0)w_2 + 2y_1y_0w_3$$

$$z_3 = (2x_3x_0 + 2x_2x_1) + (x_3y_0 + x_2y_1 + x_1y_2)v_0 + (x_2y_0 + x_1y_1 + x_0y_2)v_1 + (x_1y_0 + x_0y_1)v_2 + 2y_2y_1w_0 + (y_1^2 + 2y_2y_0)w_1 + 2y_1y_0w_2 + y_0^2w_3$$

$$z_2 = (x_1^2 + 2x_2x_0) + (x_2y_0 + x_1y_1 + x_0y_2)v_0 + (x_1y_0 + x_0y_1)v_1 + x_0y_0v_2 + (y_1^2 + 2y_2y_0)w_0 + 2y_1y_0w_1 + y_0^2w_2$$

$$z_1 = 2x_1x_0 + (x_1y_0 + x_0y_1)v_0 + x_0y_0v_1 + 2y_1y_0w_0 + y_0^2w_1$$

$$z_0 = x_0^2 + x_0y_0v_0 + y_0^2w_0$$

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