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SOME CLARIFICATIONS OF THE TUCKALS2 ALGORITHM APPLIED TO THE IDIOSCAL PROBLEM

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Kroonenberg and de Leeuw have suggested fitting the IDIOSCAL model by the TUCKALS2 algorithm for three-way components analysis. In theory, this is problematic because TUCKALS2 produces two possibly different coordinate matrices, that are useless for IDIOSCAL unless they are equal. Kroonenberg has claimed that, when IDIOSCAL is fitted by TUCKALS2, the resulting coordinate matrices will be identical. In the present paper, this claim is proven valid when the data matrices are semidefinite. However, counterexamples for indefinite matrices are also constructed, by examining the global minimum in the case where the data matrices have the same eigenvectors. Similar counterexamples have been considered by ten Berge and Kiers in the related context of CANDECOMP/PARAFAC to fit the INDSCAL model.

Key words: IDIOSCAL, TUCKALS2, CANDECOMP, PARAFAC.

Fitting the IDIOSCAL model (Carroll & Chang, 1972; Carroll & Wish, 1974) in the least squares sense amounts to minimizing the function

$$f(X, C_1, \ldots, C_m) = \sum_{i=1}^{m} \|S_i - XC_iX'\|^2 \tag{1}$$

for given symmetric $n \times n$ matrices $S_1, \ldots, S_m$. The columns of X ($n \times r$; $r \leq n$) represent group dimensions, idiosyncratically transformed by $C_1, \ldots, C_m$. Kroonenberg and de Leeuw (1980) suggested fitting the IDIOSCAL model by the TUCKALS2 (T2) algorithm, using a technique called splitting (de Leeuw & Heiser, 1982, p. 306). That is, the two appearances of X in (1) are represented by different matrices X and Y, which are optimized independently. Specifically, T2 applied to IDIOSCAL minimizes

$$g(X, Y, C_1, \ldots, C_m) = \sum_{i=1}^{m} \|S_i - XC_iY'\|^2 \tag{2}$$

without the constraint $X = Y$. This approach is only warranted if, after convergence of T2, the constraint is inactive. A claim to this effect has been stated by Kroonenberg (1983, p. 257). It is the purpose of the present paper to examine this claim from an algebraic point of view. The claim will be shown valid when $S_1, \ldots, S_m$ are semidefinite. For the case where one or more of these matrices are indefinite, counterexamples will be given, albeit of a highly restricted nature. The counterexamples coincide with counterexamples to the claim that CANDECOMP/PARAFAC applied to the INDSCAL problem must yield equal coordinate matrices in a parallel splitting problem, treated by ten Berge and Kiers (1991).

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Some Definitions and Results for IDIOSCAL

Throughout this paper it is assumed that X and Y are of full rank r. This enables one to impose the identification constraints

$$X'X = Y'Y = I_r. \tag{3}$$

By virtue of (3), the optimal C_i, for fixed X and Y, is known to be

$$C_i = X'S_iY, \tag{4}$$

(Penrose, 1956). As a result, minimizing (2) can be simplified to minimizing

$$\tilde{g}(X, Y) = \sum_i \|S_i - XX'S_iYY'\|^2, \tag{5}$$

which comes down to maximizing the function

$$h(X, Y) = \sum_{i=1}^{m} \text{tr} X'S_iYY'S_iX, \tag{6}$$

subject to (3). In fact, this is how T2 proceeds. For fixed X, the update for Y is the matrix of r principal eigenvectors of $\sum S_iYY'S_i$, and for fixed Y, the update for X is the matrix of r principal eigenvectors of $\sum S_iXX'S_i$. More efficient procedures do exist (Kroonenberg, ten Berge, Brouwer & Kiers, 1989), but do not play a role in the present paper.

Equality of X and Y, after convergence of T2, is a prerequisite for a proper IDIOSCAL solution. However, suppose that we obtain an X and a Y that differ by a rotation. That is, let $Y = XT$ for some orthonormal T. Then replacing Y by X and $C_i = X'S_iY$ by $X'S_iX$ yields the same estimates of S_i, $i = 1, \ldots, m$. Specifically, if $Y = XT$, then

$$XC_iY' = XX'S_iYY' = XX'S_iXTT'X' = XX'S_iXX' \tag{7}$$

which shows that setting $Y = X$ and $C_i = X'S_iX$ is permitted. Accordingly, we have:

Definition 1. A T2 solution has X and Y *equivalent* if $X = YT$ for some orthonormal T.

It follows that equivalence is tantamount to having $XX' = YY'$. Equivalence is closely related to symmetry of the solution:

Definition 2. A T2 solution is *symmetric* if, for $i = 1, \ldots, m$, $XC_iY' = YC_i'X'$.

Symmetry is tantamount to having $XX'S_iYY' = YY'S_iXX'$, for $i = 1, \ldots, m$, as follows from (4).

It is immediate that equivalence implies symmetry. It will be shown below (see Result 2) that the reverse is also true, under the assumption that a solution does not involve more dimensions than necessary, as pinpointed in the next definition.

Definition 3. A solution in r dimensions is *parsimonious* if there is no solution in $r - 1$ dimensions with the same value of (2).

The concept of parsimony is closely related to that of perfect fit, and the eigenvalues of $\sum_i S_i^2$, in the following manner:

Result 1. The following three statements are equivalent.

a. A (globally optimal) solution in $r + 1$ dimensions is not parsimonious.
b. There is a perfectly fitting solution in r dimensions.
c. The last $n - r$ eigenvalues of $\sum_i S_i^2$ are zero.

Proof. First, equivalence of Items b and c will be established. Using (7) and the eigendecomposition $\sum S_i^2 = K\Lambda K'$, with K an orthonormal $n \times n$ matrix, and Λ diagonal, with diagonal entries $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, we have

$$\tilde{g}(X, Y) = \sum \|S_i - XX'S_iYY'\|^2 = \sum \text{tr}S_i^2 - \text{tr}Y' \sum S_iXX'S_iY \geq \sum \text{tr}S_i^2 - \text{tr}Y' \sum S_i^2 Y \geq \text{tr}\Lambda - (\lambda_1 + \cdots + \lambda_r) = \lambda_{r+1} + \cdots + \lambda_n \geq 0. \quad (8)$$

Clearly, if $\tilde{g}(X, Y) = 0$, the last $n - r$ eigenvalues of $\sum S_i^2$ are zero. Conversely, if these last eigenvalues are zero, we have $\sum S_i^2 = K_r\Lambda_r K_r'$, with K_r containing the first r columns of K, and Λ_r the upper left $r \times r$ submatrix of Λ. It follows from Bekker (1988, Theorem 3) that, for $i = 1, \ldots, m$, $K_rK_r'S_i^2 = S_i^2$, and hence we have $K_rK_r'S_i = S_i = K_rK_r'S_iK_rK_r'$. Taking $X = Y = K_r$, a solution with $\tilde{g}(X, Y) = 0$ has been constructed in r dimensions. Equivalence of b and c has thus been proven.

It remains to prove equivalence of Items a and b. It is obvious that a solution in $r + 1$ dimensions is not parsimonious if there is a perfectly fitting solution in r dimensions. To prove the converse statement, suppose that a perfectly fitting solution in r dimensions does not exist. Then for at least one value of j, we have S_j different from $XX'S_jYY' = XC_jY'$. Therefore, the fit for S_j can be improved by constructing, from the best rank 1 approximation to $(S_j - XC_jY')$, a better fitting solution in $r + 1$ dimensions. Specifically, an additional column for X and for Y, and an additional nonzero diagonal element for C_j can be found that improve the fit for S_j, while the remaining elements needed to extend $C_1, \ldots, C_m$ are taken zero, thus leaving the fit for S_i, $i = 1, \ldots, m$, $i \neq j$, unaffected. If necessary, the extended X and Y can be replaced by columnwise orthonormal n by $r + 1$ matrices, with parallel adjustments for $C_1, \ldots, C_m$. It follows that $r + 1$ is a parsimonious number of dimensions. This completes the proof of Result 1. □

Corollary 1. If both G_y and G_x, defined as $X'\sum_i (S_iYY'S_i)X$ and $Y'\sum_i (S_iXX'S_i)Y$, respectively, are singular, the solution is not parsimonious.

Proof. It is clear from (6) that the maximum of $h(X, Y)$ can only be attained if X contains the first r eigenvectors of $\sum S_iYY'S_i$, or a rotation thereof, that may be ignored because it does not affect $h(X, Y)$. Similarly, Y must contain the first r eigenvectors of $\sum S_iXX'S_i$. If G_y is singular, then so is $\sum S_iYY'S_i$, and the same function value for $h(X, Y)$ can be obtained when X is replaced by the $n \times (r - 1)$ matrix $\tilde{X}$ that is left when column r of X is deleted. So $h(X, Y)$ equals $h(\tilde{X}, Y)$, which can be written as the trace of the matrix $\tilde{G}_x \equiv Y' \sum S_i\tilde{X}\tilde{X}'S_iY$. Because G_x equals the sum of $\tilde{G}_x$ and a positive semidefinite matrix, singularity of G_x implies singularity of $\tilde{G}_x$. Therefore, the r-th column of Y can now also be deleted without affecting $h(X, Y)$. It follows that the solution in r dimensions is not parsimonious. □

Corollary 1 makes it possible to treat symmetry and equivalence as the same concepts, when solutions are parsimonious. It has been noted above that equivalence implies symmetry. The complementary result is the following.

Result 2. Symmetry implies equivalence, for parsimonious solutions.

Proof. Suppose that, for $i = 1, \ldots, m$, symmetry holds. Then $XX'S_iYY' = YY'S_iXX'$. Postmultiplying by S_iX and summing over i yields

$$XX' \sum_i S_iYY'S_iX = XG_y = Y \sum_i Y'S_iXX'S_iX. \tag{9}$$

If G_y is nonsingular, is clear from (9) that X is in the column space of Y. If G_y is singular, but G_x is not, we have the same result upon exchanging X and Y in (9). Finally, if both G_y and G_x are singular, the solution is not parsimonious, as follows from Corollary 1. □

When T2 is applied to matrices that are positive or negative semidefinite, equivalence is guaranteed. This will be proven in the next section.

Equivalence for T2, Applied to Semidefinite Matrices

It is clear that the function h(X, Y) is not altered when S_i is replaced by $-S_i$, for any value of i. Accordingly, when $S_1, \ldots, S_m$ are semidefinite, they are understood (here and elsewhere) to be taken positive semidefinite throughout. The next result provides the key to equivalence at the maximum of h(X, Y).

Result 3. At the maximum of h(X, Y), for $S_1, \ldots, S_m$ semidefinite, we have, for $i = 1, \ldots, m$,

$$S_i^{1/2}XX'S_i^{1/2} = S_i^{1/2}YY'S_i^{1/2}. \tag{10}$$

Proof. Writing h(X, Y) as

$$[\text{Vec}(S_1^{1/2}XX'S_1^{1/2}|\cdots|S_m^{1/2}XX'S_m^{1/2})]'[\text{Vec}(S_1^{1/2}YY'S_1^{1/2}|\cdots|S_m^{1/2}YY'S_m^{1/2})] \tag{11}$$

and using the property that, for any pair of vectors $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{a}'\mathbf{b} < \max(\mathbf{a}'\mathbf{a}, \mathbf{b}'\mathbf{b})$ unless $\mathbf{a} = \mathbf{b}$, we obtain the result at once.

It can be seen from (10) that it already suffices for equivalence when one of the matrices $S_1, \ldots, S_m$ is nonsingular. Nevertheless, this assumption is not necessary, as will now be shown. □

Result 4. At the maximum of h(X, Y), for semidefinite matrices $S_1, \ldots, S_m$, equivalence is guaranteed, if the solution is parsimonious.

Proof. Define the matrices A_x and A_y as $\sum S_iXX'S_i$ and $\sum S_iYY'S_i$, respectively. It is immediate from Result 3 that $A_x = A_y \equiv A$, when the conditions of Result 4 are satisfied. Also, both X and Y must maximize h(X, Y) = tr$X'AX$ = tr$Y'AY$. It follows that X and Y must contain r principal eigenvectors of A, possibly rotated. The rank of A must be at least r, because, otherwise, both X and Y could be taken as $n \times (r - 1)$ matrices, in violation of the parsimony requirement imposed on r, see Corollary 1 above. Suppose that $XX' \neq YY'$. Then the r-th eigenvalue of A must have multiplicity 2, at least. Let $H = (\mathbf{h}_1|\cdots|\mathbf{h}_r|\mathbf{h}_{r+1})$ be a matrix of $r + 1$ dominant eigenvectors of A, with $A\mathbf{h}_r = \lambda_r\mathbf{h}_r$ and $A\mathbf{h}_{r+1} + \lambda_r\mathbf{h}_{r+1}$, and $H'H = I_{r+1}$. The maximum is also attained by taking, instead of X, the matrix X_p, defined as

$$X_p = H\begin{pmatrix} I_{r-1} & 0 \\ 0 & \mathbf{p} \end{pmatrix}, \tag{12}$$

where $\mathbf{p}$ can be any 2-vector such that $\mathbf{p}'\mathbf{p} = 1$. It follows from Result 3 that

$$S_i H \begin{pmatrix} I_{r-1} & 0 \\ 0 & \mathbf{pp}' \end{pmatrix} H' S_i = S_i Y Y' S_i, \tag{13}$$

regardless of $\mathbf{p}$, for $i = 1, \ldots, m$. Therefore, the scalar

$$\mathbf{h}_r' S_i H \begin{pmatrix} I_{r-1} & 0 \\ 0 & \mathbf{pp}' \end{pmatrix} H' S_i \mathbf{h}_r = \mathbf{h}_r' S_i Y Y' S_i \mathbf{h}_r \tag{14}$$

is invariant under the choice of $\mathbf{p}$. Define $\mathbf{f}_i$ as the vector $(\mathbf{h}_r' S_i \mathbf{h}_r | \mathbf{h}_{r+1}' S_i \mathbf{h}_r)'$. Then it follows from (14) that the scalar c, defined as $\mathbf{f}_i' \mathbf{pp}' \mathbf{f}_i$, is independent of $\mathbf{p}$. Therefore, $\mathbf{p}'(\mathbf{f}_i \mathbf{f}_i' - cI_2)\mathbf{p} = 0$ for every $\mathbf{p}$ of unit length, which implies that $\mathbf{f}_i \mathbf{f}_i'$ (of rank 0 or 1) equals cI_2 (of rank 0 or 2), whence $\mathbf{f}_i$ vanishes. From this we have $\mathbf{h}_r' S_i \mathbf{h}_r = 0$, so $S_i \mathbf{h}_r = 0$, for $i = 1, \ldots, m$, and hence $A\mathbf{h}_r = 0$. This is not possible, because it would imply a violation of parsimony of r. This completes the proof of Result 4. □

Result 4 implies that, regardless of the algorithm used, equivalence holds at maxima of h(X, Y). It follows that T2 applied to IDIOSCAL, for semidefinite matrices, cannot be disturbed by splitting problems. In fact, this is already evident from Result 3. Once we have A_x and A_y equal, the T2 algorithm updates X and Y as principal eigenvectors matrices of the same matrix A, which guarantees that X and Y will be equal. The results obtained above have a wider generality, in that they apply to the maxima of the object function of T2, rather than to specific algorithms.

Having dealt with a class of data matrices where T2 applied to IDIOSCAL must yield a proper equivalent solution, we now turn to a class of matrices where equivalence is not granted.

The Global Minimum for IDIOSCAL in the Equal Eigenvectors Case

When $S_1, \ldots, S_m$ all have the same eigenvectors, with eigenvalues in arbitrary orders, the global minimum of (2) can be found explicitly. More importantly, necessary conditions for the global optimality of a T2 solution, relevant for symmetry, can be given. These conditions indicate very specifically how asymmetry may come about.

If $S_1, \ldots, S_m$ have the same eigenvectors, then we can write $S_i = K\Lambda_i K'$, for $i = 1, \ldots, m$, with K orthonormal and Λ_i diagonal. Accordingly, (2) can be rewritten as

$$\mathrm{g}(X, Y, C_1, \ldots, C_m) = \sum_{i=1}^{m} \|K\Lambda_i K' - XC_i Y'\|^2 = \sum_{i=1}^{m} \|\Lambda_i - K'XC_i Y'K\|^2, \tag{15}$$

which is the same least squares function, with S_i replaced by a diagonal matrix Λ_i. For reasons of simplicity, we shall drop the transformation K and examine the minimum of (2) with $S_i = \Lambda_i$ (diagonal) directly.

For fixed diagonal matrices $\Lambda_1, \ldots, \Lambda_m$ we have

$$\mathrm{g}(X, Y, C_1, \ldots, C_m) = \sum_{i=1}^{m} \|\Lambda_i - XC_i Y'\|^2$$

$$\geq \sum_{i=1}^{m} \|\Lambda_i - X(X'X)^{-1}X'\Lambda_i\|^2$$

$$= \sum_{i=1}^{m} \|\Lambda_i - XX'\Lambda_i\|^2$$

$$= \sum_{i=1}^{m} \mathrm{tr}\Lambda_i^2 - \mathrm{tr}P_x \sum_{i=1}^{m} \Lambda_i^2, \tag{16}$$

with $P_x = XX'$, an idempotent matrix. The rationale underlying (16) is that $C_i Y'$ cannot outperform the linear regression weights for estimating Λ_i from X, $i = 1, \ldots, m$. Let $\Lambda_1, \ldots, \Lambda_m$ be permuted such that $\sum \Lambda_i^2$ has its diagonal elements in weakly decreasing order. Then we have from ten Berge (1983) that

$$\mathrm{tr}P_x \sum \Lambda_i^2 \leq \mathrm{tr}E_r \sum \Lambda_i^2, \tag{17}$$

where E_r is the $n \times n$ diagonal matrix with r unit elements, followed by $n - r$ zero elements on the diagonal. Combining (16) and (17) gives the lower bound

$$g(X, Y, C_1, \ldots, C_m) \geq \sum \mathrm{tr}\Lambda_i^2 - \mathrm{tr}E_r \sum \Lambda_i^2. \tag{18}$$

The lower bound is in fact the global minimum because it can be attained. That is, if we let

$$X = Y = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \text{ and } C_i = X'\Lambda_i Y, \tag{19}$$

$i = 1, \ldots, m$, (16) holds as an equality, with $X'X = Y'Y = I_r$. Clearly, (19) is a solution which has the property of symmetry and equivalence. We shall now examine under what conditions asymmetry and hence nonequivalence may occur, at the global minimum.

If a T2 solution is to be globally minimal, then both (16) and (17) must hold as equalities. That is, the solution must satisfy

$$C_i Y' = X'\Lambda_i, \tag{20}$$

$i = 1, \ldots, m$, and

$$\mathrm{tr}P_x \sum \Lambda_i^2 = \mathrm{tr}E_r \sum \Lambda_i^2. \tag{21}$$

Repeating the derivation of (16) and (17) with the roles of X and Y interchanged yields the further condition, parallel to (20), that

$$C_i'X' = Y'\Lambda_i, \tag{22}$$

$i = 1, \ldots, m$. Clearly, if the r-th and $r + 1$-th element of $\sum \Lambda_i^2$ are distinct, then $P_x = E_r$ is the only solution for (21), and hence, for $i = 1, \ldots, m$,

$$XC_i Y' = P_x\Lambda_i = E_r\Lambda_i = \Lambda_i E_r = \Lambda_i P_x = YC_i'X', \tag{23}$$

which establishes symmetry. However, if this distinctness condition is not met, we may or may not have symmetry. A more general condition can be obtained from (20) and (22). Combining these, we have, for $i = 1, \ldots, m$,

$$XC_i Y' = P_x\Lambda_i = \Lambda_i P_y \tag{24}$$

with $P_y = YY'$, and hence

$$\Lambda_i^2 P_x = \Lambda_i(\Lambda_i P_x) = \Lambda_i P_y \Lambda_i = P_x \Lambda_i \Lambda_i = P_x \Lambda_i^2, \tag{25}$$

which shows that P_x commutes with Λ_i^2. Symmetry, on the other hand, can be redefined in the present context as having P_x commuting with Λ_i, $i = 1, \ldots, m$, see (24). The question is, under what condition is it possible that P_x commutes with Λ_i^2 without commuting with Λ_i. The very same question has been examined by ten Berge and Kiers (1991, pp. 320–322) in a related context. They have shown that this is possible if Λ_i has at least two diagonal elements that differ in sign only, one of which is among the first r elements of Λ_i, while the other is not. In the present context, we have to deal with $\Lambda_1, \ldots, \Lambda_m$ simultaneously, which is quite a bit more involved. Let L be defined as the $m \times r$ matrix containing the diagonal elements of Λ_i in its i-th row, $i = 1, \ldots, m$.

Result 5. At the global minimum of (2) in the equal eigenvectors case symmetry is guaranteed unless L can be arranged to have an r-th column $\mathbf{l}_r$ and an $r + 1$-th column $\mathbf{l}_{r+1}$ equal to $-\mathbf{l}_r$, the freedom to rearrange columns in L being limited by the requirement that the column sums of squares, that is, the diagonal elements of $\underline{\sum} \Lambda_i^2$, must be in weakly descending order.

Proof. It is convenient to first reduce the order of the matrices involved, without affecting any essential property of the relevant equations. Let $\underline{\sum} \Lambda_i^2$ be denoted as Λ, and let Λ be partitioned as

$$\Lambda = \begin{pmatrix} \Lambda_s & 0 & 0 \\ 0 & \Lambda_t & 0 \\ 0 & 0 & \Lambda_u \end{pmatrix}, \tag{26}$$

with t the multiplicity of the r-th diagonal element, $s \le r \le (s + t)$, and $u = n - s - t$. At the global minimum of (2), we can infer from (21) that P_x is also a block-diagonal matrix of the same dimensions as Λ, with as a first block the $s \times s$ identity matrix I_s, as a second block a symmetric and idempotent $t \times t$ matrix P_{xt} of rank $r - s$, and zeroes elsewhere, including the last u rows and the last u columns. It follows that symmetry holds for XC_iY' if and only if P_{xt} commutes with Λ_{it}, the corresponding diagonal $t \times t$ submatrix of Λ_i. Thus having reduced the problem to one concerning submatrices, we now consider various special cases.

First, suppose that Λ_t is a zero matrix. Then Λ_{it} is also zero, for $i = 1, \ldots, m$, and symmetry is obvious.

Next, let Λ_t be nonzero, and define L_t as the $m \times t$ submatrix of L, containing the diagonal elements of $\Lambda_{1t}, \ldots, \Lambda_{mt}$. If $t = 1$, symmetry is immediate, because the r-th and $r + 1$-th element of Λ are distinct. So let $t > 1$. Now choose a value of i such that Λ_{it} is nonsingular. In case such a Λ_{it} does not exist, replace it by any nonsingular linear combination of $\Lambda_{1t}, \ldots, \Lambda_{mt}$. Next, choose a value of $j \neq i$ such that Δ, defined as $\Lambda_{jt}\Lambda_{it}^{-1}$, has no pair of equal diagonal values. These two steps can always be taken unless L_t has two proportional columns, a case to be treated separately. Noting that (24) remains valid when C_i and Λ_i are replaced by identically weighted linear combinations, we have

$$P_{xt}\Lambda_{jt} = \Lambda_{jt}P_{yt} \text{ and } P_{xt} = \Lambda_{it}P_{yt}\Lambda_{it}^{-1}, \tag{27}$$

with P_{yt} defined analogously to P_{xt}. It is clear from (27) that

$$\Lambda_{it}P_{yt}\Lambda_{it}^{-1}\Lambda_{jt} = \Lambda_{jt}P_{yt}, \tag{28}$$

which shows that $P_{yt}\Delta = \Delta P_{yt}$. All diagonal elements of Δ being distinct, it follows that P_{yt} is a diagonal matrix. Hence P_y is diagonal, whence symmetry is obtained, see (24).

Finally, we need to consider the case where L_t has proportional columns. These columns are either identical or they differ in sign only. If they are identical, the multiplicities of the diagonal elements of Λ_t are the same as those for Λ_{it}, $i = 1, \ldots, m$. As a result, Λ_t commutes with the same matrices as does Λ_{it}, $i = 1, \ldots, m$. Applying this to (24) yields symmetry. The only case that remains is that where Λ_t has two columns that differ only in signs. In this case asymmetry can always be obtained, by taking P_{xt} nondiagonal. □

To facilitate the interpretation of Result 5, a counterexample where the conditions of Result 5 are not met will be instructive. Let

$$\Lambda_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ and } \Lambda_2 = \begin{pmatrix} .5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{29}$$

and let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{.5} \\ 0 & \sqrt{.5} \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{.5} \\ 0 & -\sqrt{.5} \end{pmatrix}. \tag{30}$$

Then we have

$$XC_1Y' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \text{ and } XC_2Y' = \begin{pmatrix} .5 & 0 & 0 \\ 0 & .5 & -.5 \\ 0 & .5 & -.5 \end{pmatrix}. \tag{31}$$

This solution attains the global minimum 5 of (2), yet symmetry is lacking. Another counterexample can be found in ten Berge and Kiers (1991, p. 322). Although that was derived to show that CANDECOMP/PARAFAC applied to INDSCAL need not yield symmetry at the global minimum, it also pertains to T2 applied to IDIOSCAL. This is not a coincidence, but follows from the fact that Result 5 and the derivations preceding it also apply to INDSCAL, upon redefining P_x and P_y as $X(X'X)^{-1}X'$ and $Y(Y'Y)^{-1}Y'$, respectively.

Above, it was shown that symmetry is guaranteed in the case of semidefinite matrices $S_1, \ldots, S_m$. Clearly, matrices, with a pair of eigenvalues that differ in signs are indefinite, which reconciles Result 5 with Result 4. Also, it was shown that having the r-th and the $r + 1$-th diagonal elements of $\overline{\sum \Lambda_i^2}$ distinct implies symmetry in the equal eigenvectors case. Clearly, the counterexample of (29) and (30) violates this condition.

Discussion

Comparing the theoretical results for splitting in IDIOSCAL with those for INDSCAL (ten Berge & Kiers, 1991), it can be seen that the situation is much better for IDIOSCAL than for INDSCAL, because the case of semidefinite matrices has been solved for IDIOSCAL (Result 4), whereas a parallel result for INDSCAL is lacking. In fact, if such a result for INDSCAL will ever be obtained, it will only pertain to global minima, because cases of asymmetry at *local* minima in INDSCAL, applied to semidefinite matrices, have already been encountered (ten Berge & Kiers, 1991, p. 324).

Practical experience with T2 applied to IDIOSCAL seems to give no problems with asymmetry whatsoever. The positive Result 4 of this paper, and the observation that counterexamples for indefinite matrices could only be constructed in highly artificial circumstances, should further strengthen our confidence in T2 as a suitable method for IDIOSCAL.

Kiers (1989) has developed an alternative IDIOSCAL algorithm which preserves equivalence throughout the computations. In view of the efficiency of T2, and its theoretical and practical properties in the application to IDIOSCAL, there seems to be no need to abandon T2 in favor of such an equivalence preserving algorithm.

Kiers, Cléroux and ten Berge (in press) considered an application of IDIOSCAL in the context of optimal matrix correlations. In this context, the symmetric matrices $S_1, \ldots, S_m$ to be analyzed are Gramian. The latter property enabled Kiers et al. to develop a monotonically convergent algorithm that preserves equivalence throughout the computations, as an alternative to TUCKALS2. Result 4 of the present paper implies that there was no need to develop such an algorithm because TUCKALS2, applied to Gramian matrices, cannot be disturbed by splitting problems.

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