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DETERMINATION OF PARAMETERS OF A FUNCTIONAL RELATION BY FACTOR ANALYSIS*

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Consideration is given to determination of parameters of a functional relation between two variables by the means of factor analysis techniques. If the function can be separated into a sum of products of functions of the individual parameters and corresponding functions of the independent variable, particular values of the functions of the parameters and of the functions of the independent variables might be found by factor analysis. Otherwise approximate solutions may be determined. These solutions may represent important results from experimental investigations.

The possible use of factor analysis techniques to determine parameters of nonlinear functional relations has been a topic for occasional informal discussion. If a factorial approach could be developed it would have considerable application to experimental problems such as learning curves, work decrement curves, dark adaptation curves, etc. This note gives a theoretical basis for determination of parameters by factor analysis for many nonlinear functions.

Factor analytic methods have been limited to investigations applying linear functions of the form (see [2], equation 3, p. 71):

$$s_{ji} = \sum_{m=1}^{r} a_{jm} s_{mi} , \tag{1}$$

where the s_{ji} are the observations, and a_{jm} and s_{mi} are to be estimated. The a_{jm} are task parameters, and the s_{mi} are individual parameters.

In the present context we will consider the functional relation between two variables x and y. Variable x might be termed the independent variable and y might be termed the dependent variable. A general statement of this functional relation for any given individual i is given by

$$y_i = \phi(p_{oi} , x), \tag{2}$$

for which there are a number of parameters p_o which have specific values

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p_{gi} for each individual. Such a relation is shown graphically in Fig. 1. There exists a family of functions of the form of any given ϕ with the values of p_{gi} defining the particular member of the family. Let j be a particular point of this function with coordinates x_j and y_{ji} . Then

$$y_{ji} = \phi(p_{gi}\ , x_j). \tag{3}$$

Many functions may be transformed so as to produce

$$y_{ji} = \sum_{m=1}^{r} f_m(x_j)F_m(p_{gi}). \tag{4}$$

The $f_m(x_j)$ are a number of functions of the independent variable x_j . The $F_m(p_{gi})$ are corresponding functions of the parameters p_{gi} . The number, r, of such functions may be finite, or it may be infinite. In this latter case, (4) represents an infinite series, such as Maclaurin's or Taylor's power series or Fourier's trigonometric series (see a standard advanced calculus text, e.g.,

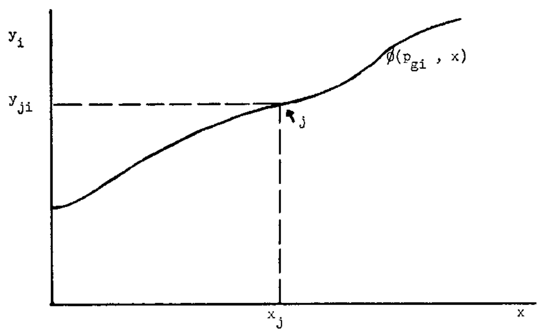

FIGURE 1
A Functional Relation of the Form of (2)

[1], [3]). Frequently, in this case, a small number of terms of the series will yield an adequate approximation to the y_{ji} . In order to make (1) applicable it is only necessary to define

$$a_{jm} \equiv f_m(x_j), \tag{5}$$

$$s_{mi} \equiv F_m(p_{gi}). \tag{6}$$

Then

$$y_{ji} = \sum_{m=1}^{r} a_{jm}s_{mi}\ . \tag{7}$$

In the present context the s_{mi} will be considered as derived parameters of the transformed function. While they may be expressible in terms of more primitive parameters, they do have the property of determining the particular function for each individual. The family of functions is defined by the a_{jm}. As a consequence of (7), observations of y_{ji} for several given x_j and individuals i may be entered into a score matrix. Each x_j might be used to produce one statistical variable. Estimates of the a_{jm} and s_{mi} then can be obtained by factor analysis techniques.

In order to illustrate the foregoing, consider a learning task for which the learning curve is a simple exponential function, such as

$$(8) \qquad y_{ji} = e^{(t_j+b_i)},$$

where y_{ji} is the performance of individual i on trial j, b_i is a parameter for individual i, and t_j is the number of trials j. t_j replaces x_j as the independent variable in this context, and b_i replaces the parameters p_{gi}. Equation (8) may be transformed to

$$(9) \qquad y_{ji} = (e^{t_j})(e^{b_i}).$$

Then

$$(10) \qquad a_{j1} = f_1(t_j) = e^{t_j},$$

$$(11) \qquad s_{1i} = F_1(b_i) = e^{b_i}.$$

In this case only one term of the sum of products indicated in (4) and (7) exists. From (9), (10), and (11)

$$(12) \qquad y_{ji} = a_{j1}s_{1i}.$$

For this simple case, observations are made of the performances on the learning task for each of a number of individuals at each of a selected number of trials. These observations yield a matrix of y_{ji}. A factor analysis will involve a single factor and yield estimates of the a_{j1} and s_{1i}.

The factor analysis problems of communalities and rotation of axes remain to be discussed. In the present context it seems appropriate to assume that each observed y_{ji} may be in error, but the assumption of specific factors seems inappropriate. As a consequence, reliability estimates should be placed in the diagonals of the matrix of intercorrelations. The rotation of axes problem remains unsolved in the present case. The solution is not unique, and the axes may be rotated. It is doubtful, moreover, that the principle of simple structure is applicable when the factor loadings are the various values of the functions $f_m(x_j)$ for the selected points. Some other principle, at present unknown, is needed to fix the location of the axes.

An alternative interpretation of (7) corresponds to the obverse factor procedures, where people are correlated over a population of measures. A large number of values of x_j are selected, and the y_{ji} are observed for a group of individuals. Each of these individuals can be considered as a variable and correlations of the y_{ji} can be obtained for pairs of individuals. The s_{mi} are now the factor loadings, and the a_{jm} are the factor scores. The communalities and rotation of axes aspects of the analysis are quite similar to the corresponding aspects of the first procedure already discussed. One important difference between the present analysis by persons and the previous alternative stems from the more direct determination of the s_{mi}. An inspection of the matrix of s_{mi} might reveal a curvilinear relation between the s_{mi} for several m. Any such relation as the entries in one row being proportional to the square of the entries in another row would indicate a relation to a common, more primitive parameter. The entries in one row being proportional to the product of corresponding entries in two other rows would also be indicative of more primitive parameters. Rotation of axes might be performed so as to reveal such relations.

In any particular situation, the choice as to which variable is to be the independent variable x and which variable is to be the dependent variable y may be quite important. In a learning experiment for a list of paired associates, each trial might be an x_j, and the proportion of correct responses be the observed y_{ji}. However, selected proportions of correct responses might be taken as the x_j, and the numbers of trials necessary to reach these proportions taken as the y_{ji}. Consider a slightly more complex exponential learning curve than that given in (8), such that

$$P = e^{(c_i t + b_i)}, \tag{13}$$

where P is the measure of performance. The parameter c_i has been included as a multiplier to t. This function does not separate in the manner that (8) did unless an infinite series is used. In which case, if values of t_j are chosen and values of P_{ji} are observed, the factor analysis will not involve a definite number of factors. Each successive factor will permit a closer approximation of the series to the function. Some finite number of factors might be found to be adequate.

If logarithms are taken of both sides of (13), it is possible to solve for t as a function of P:

$$t = \frac{1}{c_i} \log P + \frac{b_i}{c_i}. \tag{14}$$

When values of P are selected as P_j and the corresponding t_{ji} are observed, then

$$t_{ji} = \frac{1}{c_i} \log P_j + \frac{b_i}{c_i}. \tag{15}$$

Define

$$a_{j1} \equiv \log P_j \, , \tag{16}$$

$$s_{1i} \equiv 1/c_i \, , \tag{17}$$

$$a_{j2} \equiv 1, \tag{18}$$

$$s_{2i} \equiv b_i/c_i \, . \tag{19}$$

Then

(20) $$t_{ji} = a_{j1}s_{1i} + a_{j2}s_{2i} \ ,$$

which is in the form of (7). Only two factors are involved.

Another extension from (8) is to introduce an additive constant d_i:

(21) $$P = d_i + e^{(t+b_i)}.$$

Individual parameters and the variable t may be separated for (21) in the same manner as given for (8). There are now two factors.

If both of the foregoing extensions of (8) are incorporated into a single extension, then

(22) $$P = d_i + e^{(c_i t+b_i)}.$$

The individual parameters do not readily separate now from either variable without employing an infinite series.

It is to be noted that (8) might be treated in the same manner as was (13). The individual parameters might be separated from the variable y or P rather than from t as given. Thus, the foregoing examples include (*i*) a function, equation (8), that may be treated either way; (*ii*) two functions, (13) and (21), each of which may be treated in only one manner; and (*iii*) a function, (22), that cannot be separated. The two single treatment functions form a contrast as to which variable, P or t, is taken as the independent variable. In (13), P should be taken as the independent variable while in (21) t should be taken as the independent variable. In any particular experimental case, the decision as to which variable is to be treated as the independent variable must rest on experience and the judgment of the experimenter. There are cases where the number of factors is excessive whichever variable is taken as the independent variable. The factorial approach may yield in some of these cases an adequate approximation to the observations with a limited number of factors.

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