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AN INDIVIDUAL DIFFERENCES MODEL FOR MULTIDIMENSIONAL SCALING*

LEDYARD R TUCKER

UNIVERSITY OF ILLINOIS

AND

SAMUEL MESSICK†

EDUCATIONAL TESTING SERVICE

A quantitative system is presented to permit the determination of separate multidimensional perceptual spaces for individuals having different viewpoints about stimulus interrelationships. The structure of individual differences in the perception of stimulus relationships is also determined to provide a framework for ascertaining the varieties of consistent individual viewpoints and their relationships with other variables.

The present paper attempts to develop a quantitative system to provide for differential representations of perceptual structures for individuals having different viewpoints about stimulus interrelationships. In past attempts at stimulus scaling, two major approaches have been employed in dealing with data obtained from groups of individuals. One approach has been to ascertain group averages and then to generalize findings to the "average person" in each group; the other procedure has been to work with each person separately and to enumerate the results individual by individual. The first method, which is the more usual in practice, may lead to a straightforward but possibly false interpretation, in that the results for the average person may not describe very accurately the consistent responses of each individual in the sample [16a]. The second method of working with each individual separately also possesses several drawbacks, among which are the extensiveness of experimental observations required to obtain stable results for each individual and the difficulties involved both in describing the results for groups of individuals and in comparing the results for several individuals and groups.

The intention in the present paper is to develop a system which will

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provide not only multidimensional descriptions of individual perceptual structures and a basis for comparisons between individuals and groups, but also a superstructure to represent the varieties or types of consistent individual perceptions. The present model is thus concerned both with the multidimensional scaling of stimuli and with the structure of consistent individual differences in perception and judgment.

Many of the multidimensional scaling methods based upon an averaging of responses over all individuals in a sample, either in terms of simple averages as in the method of equal-appearing intervals or in terms of more complicated scaling functions as in the methods of successive intervals and of complete triads, have been reviewed by Messick [25] and Torgerson [45]. The application of this multidimensional scaling of the "average individual" in a group presents certain difficulties, however, when comparisons are attempted between perceptual structures obtained from diverse groups that presumably have different orientations to the stimuli. A common finding has been that only subtle differences appear in these structures and that the main attributes of the perceived spaces are essentially identical (Abelson [1], Messick [24, 28]). It may be that in these studies all individuals perceived the stimulus interrelations in more or less the same manner, thus yielding the obtained observations of only minor differences between groups. On the other hand, it may be that extensive differences existed in individual perceptual spaces, but the scaling method blended them together in deriving the average structure for each group. We might not have discovered yet how to sort individuals into contrasting groups that would have different perceptual structures for their average persons. The variables employed so far for establishing such comparison groups may be only slightly related to individual differences in perceptual structures.

It would be desirable, then, to develop a procedure for uncovering differential perceptual spaces that does not require prior sorting of individuals into subgroups on the basis of variables presumed to differentiate between perceptual structures, but one that would instead indicate the variety of individual perceptual structures represented in the total group. A technique is thereby required that would first isolate empirically any consistent individual viewpoints about stimulus differences and would then provide for the derivation of separate multidimensional spaces for each viewpoint.

In the present discussion, the multidimensional scaling model developed by Richardson [36], using the Young and Householder [49] theorems, and extended by Torgerson [44], Messick and Abelson [29], and Shepard [40, 41] will form the basis for description of the perceptual structure for each individual. In this model, each stimulus is represented by a point in a Euclidean space, with the perceived difference or dissimilarity between two stimuli represented by the distance between the two stimulus points. Measures of perceived dissimilarity among stimuli may be obtained by several experimental procedures (cf. Shepard [39]), such as judging which of two stimuli is more similar to a third stimulus (Richardson [36], Torgerson [43]), rating the dissimilarity between members of pairs of stimuli on some rating scale (Attneave [4], Abelson [1], Ekman [10], Messick [26], Jackson, Messick, and Solley [22], Coombs [7], Reeb [35], Abelson and Sermat [2]), ranking the similarity of the remaining stimuli to each stimulus in turn (Klingberg [23], Morton [32]), using intrusion errors in identification learning as measures of proximity (Shepard [38]), and estimating interstimulus distance directly on a ratio scale (Helm [15], Indow and Kanazawa [19], Indow and Uchizono [20]). Various refinements in analysis are possible, including the application of scaling techniques to recover interval properties for the dissimilarity or distance scale (cf. Adams and Messick [3], Torgerson [45]) and solution for an additive constant to establish an advantageous scale origin to yield the simplest multidimensional representation (Messick and Abelson [29], Torgerson [45]). The multidimensional perceptual space can then be derived by factor analyzing scalar products between stimulus vectors computed from the distance estimates (Young and Householder [49], Torgerson [45]) or by applying Shepard's [40] computer model for constructing a Euclidean metric configuration directly from nonmetric proximity information.

The problem of uncovering and representing consistent individual viewpoints about stimulus properties has been addressed for the case of unidimensional scales by Tucker [46, 47]. Tucker [46] developed a vector model for paired comparisons that permits judges, when evaluating stimuli with respect to some unidimensional attribute or in terms of preference, to differ among themselves in their perceived ordering and spacing of the stimuli. Each individual viewpoint is represented as a vector in a multidimensional space of stimulus objects, with stimulus projections on each vector representing scale values for that viewpoint. The number of dimensions required to span the space of individual viewpoint vectors is determined by factor analysis, and the resulting rotated factor loadings represent stimulus scale values for the various viewpoints. Slater [41a] also suggested the use of principal components to analyze covariation in preferences, and a similar rationale underlies the factor analysis of category ratings (Morris and Jones [31], Messick [27]). Bock [5a] employed the closely related procedures of discriminant analysis to select judges reflecting the same dimension of preference; he also suggested the use of subsequent canonical vectors to estimate additional significant preference dimensions if any were found.

The dimensions isolated in these vector models summarize consistencies in ratings of separate stimuli with respect to some specified attribute, and they represent consistent individual viewpoints about that stimulus property. The dimensions in the distance model of multidimensional scaling, on the other hand, are derived from judgments about pairs of stimuli with respect to similarity, and they represent differential attributes of perceived stimulus variation. In the vector model, then, the multidimensional space represents the different viewpoints of the judges, each viewpoint being a one-dimensional scale of the specified stimulus attribute. In the distance model, the multidimensional space represents the different ways in which the stimuli are perceived to vary, each judge perceiving the space in essentially the same manner. The present paper attempts to combine these two approaches by applying the vector model of stimulus scaling to measures of similarity between pairs of stimuli, thereby isolating dimensions of individual viewpoints about stimulus similarity or proximity. Stimulus projections (in this case for pairs of stimuli) on each rotated dimension of viewpoint will then provide measures of similarity to be analyzed according to the distance model of multidimensional scaling (Messick and Abelson [29], Torgerson [45], Shepard [40]). A separate multidimensional representation of the perceived stimulus space is thus provided for each consistent viewpoint about stimulus similarity.

A nontechnical discussion of these methods and their development was presented in the context of social perception by Jackson and Messick [21], and some recent applications were described by Gulliksen [11, 12] and Tucker [48]. Helm and Tucker [16] applied these methods to color perception.

Analysis of Consistent Individual Viewpoints in Multidimensional Scaling

The present model assumes that estimates of interstimulus distances are available for each individual. As indicated previously, these estimates may be obtained experimentally by several procedures, such as rating the dissimilarity between stimuli on a rating scale or constructing the interstimulus distances directly by ratio judgments.

Let $x_{(jk)i}$ = an estimate of dissimilarity or interpoint distance between stimuli j and k by individual i;
i, h = individuals $1, 2, \cdots, N$;
j, k = stimuli $1, 2, \cdots, n$;
(jk) = stimulus-pairs 12, 13, 23, etc.; $k > j$; number of stimulus-pairs $= n(n-1)/2$.

There is one such distance measure for each pair of stimuli and each individual, so that these measures may be arrayed in a rectangular table with a row for each pair of stimuli and a column for each individual. All cells in this table (designated matrix X) should be filled; i.e., there should be no missing data.

X = matrix of $x_{(jk)i}$, having $n(n-1)/2$ rows for the stimulus-pairs and N columns for the individuals.

The typical multidimensional scaling analysis (cf. Torgerson [45]) involves an averaging, frequently weighted in terms of a scaling function, of the $x_{(jk)i}$ values over the individuals to obtain a single number or scale value to represent the dissimilarity or distance between each pair of stimuli j and k. These averaged or scaled distance values are then usually analyzed according to the Young-Householder theorems to obtain a multidimensional representation (Messick and Abelson [29]; Torgerson [45]). This procedure assumes that these distance measures adequately summarize the information in the distribution of $x_{(jk)i}$ values over the i individuals and that the variation in these values is due to random dispersion or error of measurement. The present analysis, on the other hand, first asks whether there is consistent covariation among individuals in these $x_{(jk)i}$ estimates by factoring X into its principal components. If only one factor is found to account for the consistent variance in X, then the appropriate factor loadings, or other types of average distance values, may be analyzed as usual to obtain a single representative multidimensional space. If, on the other hand, more than one factor is necessary to account for the variance in X, then more than one set of distance values will be obtained from the factor loadings to be subsequently analyzed by multidimensional scaling procedures. Several multidimensional spaces would thereby be derived representing different points of view about the perceived stimulus arrangements. This analysis of dissimilarity estimates parallels the general argument outlined by Holzinger [16a] for the complete factor analysis of scores as an alternative to the incomplete summarization of data provided by a single average when the rank of the score matrix exceeds unity.

In this procedure, dimensions of viewpoint are obtained for consistent individual differences in the dissimilarity or distance estimates. Since there will presumably be fewer consistent viewpoints than there are individuals, the technique appears more efficient than analyzing each individual's distance estimates separately. Also, as will be seen below, the present method provides a framework for comparing the various viewpoints and for relating them to outside variables.

The central point in the above discussion was the statement that consistent covariation in the $x_{(jk)i}$ estimates is evaluated by factoring X into its principal components. Since X is an asymmetric, rectangular matrix, however, the usual direct factoring equations are not appropriate (e.g., Harman [13]). This problem has been solved by factoring X according to a theorem of Eckart and Young [9].

Determining Dimensions of Individual Differences

Since the number of stimulus-pairs is related to the square of the number of stimuli, the number of rows in matrix X is likely to be relatively large. If 20 stimuli were used, for example, the number of stimulus-pairs would be 190; for 25 stimuli there would be 300 pairs. Consequently, it is advantageous to use a moderately small sample of individuals and to perform a type of obverse analysis in which relationships are computed between individuals rather than variables. The basic matrix in this analysis, designated matrix P, is composed of sums of squares of measures for the individuals in the main diagonal and sums of cross products of measures between pairs of individuals off diagonal, all sums being taken over the pairs of stimuli. Thus, in terms of matrix algebra,

$$P = X'X, \tag{1}$$

an $N \times N$ matrix of sums of cross products between columns of X.

The analysis next follows the procedure developed by Eckart and Young [9] to obtain a matrix $\hat{X}$ of lower rank than matrix X that approximates X in a least-squares sense. This analysis parallels Horst's development [17, pp. 364-382]. Essentially, the matrix $\hat{X}$ is constructed to the desired degree of approximation from the r largest characteristic roots and vectors of matrix X.

$$\hat{X}_r = U_r \Gamma_r W_r , \tag{2}$$

a matrix of rank r that approximates matrix X in a least-squares sense [9], where

$U_r = n(n - 1)/2 \times r$ section of an orthogonal matrix ($U_r' U_r = I$),
$\Gamma_r = r \times r$ diagonal matrix of latent roots,
$W_r = r \times N$ section of an orthogonal matrix ($W_r W_r' = I$).

This analysis is similar to the principal-components method developed by Hotelling [18], but differs in that the components are derived from the matrix of sums of squares and cross products of raw measures instead of from a matrix of intercovariances as in the Hotelling procedure (cf. Nunnally [34]). The components U_r, Γ_r, and W_r in the basic Eckart-Young theorem (2) are determined from the characteristic roots and vectors of the cross-products matrix P. Since P, unlike X, is a square, symmetric matrix, it may be analyzed directly into principal components by standard procedures [13].

$$\hat{P}_r = \hat{X}_r' \hat{X}_r = W_r' \Gamma_r^2 W_r , \tag{3}$$

where Γ_r^2 is a diagonal matrix composed of the r largest characteristic roots of P, and W_r contains, as row vectors, the corresponding characteristic vectors of P. Note that the characteristic roots of P are the squares of the diagonal entries in matrix Γ_r, so that the diagonal matrix Γ_r in (2) must be constructed from the square roots of the values in Γ_r^2 from (3).

The matrix U_r may now be computed by

$$U_r = X W_r' \Gamma_r^{-1}, \tag{4}$$

since $W_r W_r' = U_r' U_r = I$.

If in some experiments the number of individuals is greater than the number of stimulus-pairs, the cross-products matrix of (1) should instead be computed between stimulus-pairs, summing over individuals. The cross-products matrix in the present analysis should always be computed between the variables on the shorter side of X, summing over the variables on the longer side. Thus, if $N > n(n-1)/2$,

$$P = XX', \tag{1a}$$

an $n(n-1)/2$ by $n(n-1)/2$ matrix of sums of cross products between rows of X. The remainder of the analysis follows by symmetry:

$$\hat{X}_r = U_r \Gamma_r W_r , \tag{2a}$$

$$\hat{P}_r = \hat{X}_r \hat{X}_r' = U_r \Gamma_r^2 U_r' , \tag{3a}$$

$$W_r = \Gamma_r^{-1} U_r' X. \tag{4a}$$

The elements in W_r' represent projections of points corresponding to individuals on unit-length principal vectors of X (and P). The elements in U_r represent projections of points corresponding to stimulus-pairs on unit-length principal vectors of X. These stimulus-pair projections, when appropriately weighted, scaled, and rotated to orientations possibly more appropriate psychologically than the principal-axes position, will constitute measures of distance between pairs of stimuli. There will be at least as many sets of distance measures as there are columns in the U_r matrix, each set being subsequently analyzed by multidimensional scaling procedures.

Scaling for Differences in Sample Size

The above analysis produces coefficients for stimulus-pairs and for individuals that are scaled so that $W_r W_r' = I$. Since W_r is a matrix of order r by N, the resulting coefficients are a function of the number N of individuals in the sample. Thus, even if two multidimensional scaling studies differed only in sample size, i.e., if the same stimuli were involved and the judges consisted of two random samples of different size from the same population of individuals, the resulting numbers would not be comparable. It is desirable, then, to rescale W_r into a matrix V:

$$V = KW_r , \tag{5}$$

so that the coefficients in V are independent of sample size; i.e.,

$$\frac{1}{N} VV' = I, \tag{6}$$

where K and $1/N$ are scalar matrices with diagonal elements K and $1/N$, respectively. Substituting (5) into (6) and solving,

$$K = N^{1/2}, \tag{7}$$

$$V = N^{1/2}W_r \,. \tag{8}$$

To maintain the basic relationship of (2), U_r must then be rescaled to

$$Y = U_r N^{-1/2}. \tag{9}$$

Thus, from (2)

$$\hat{X}_r = Y\Gamma_r V = U_r N^{-1/2}\Gamma_r N^{1/2} W_r = U_r \Gamma_r W_r \,, \tag{10}$$

since $N^{1/2}$ and $N^{-1/2}$ are scalar matrices.

The matrix Y now contains scaled stimulus-pair projections on the principal vectors, and the matrix V contains, as row vectors, scaled individual projections on the principal vectors. From (4),

$$Y = XV'\Gamma_r^{-1}N^{-1}. \tag{11}$$

The V matrix of scaled projections of individuals on principal vectors may be converted into a factor matrix A of scaled projections of individuals on principal factors by weighting each vector by the square root of the corresponding characteristic root:

$$A = \Gamma_r V = N^{1/2}\Gamma_r W_r \,. \tag{12}$$

Then, from (10),

$$\hat{X}_r = YA. \tag{13}$$

Rotation to Structure in the Space of Individuals

Since the principal-axes location may not be the most appropriate orientation for dimensions of viewpoint about stimulus similarity, a rotation of the obtained A and Y matrices might be considered. This possibility is analogous to the rotation of axes in factor analysis. One criterion for such a rotation would be a search for simple structure, by either graphical or analytical procedures [13], in the factor space of the individuals.

An r by r nonsingular transformation matrix T is sought to rotate these principal factors to simple structure or some other criterion:

$$B = TA. \tag{14}$$

A matrix Z of scaled stimulus-pair projections on these rotated axes is obtained by

$$Z = YT^{-1}. \tag{15}$$

It should be noted that the basic condition of the Eckart-Young theorem in (2) is still satisfied under these transformations:

$$\hat{X}_r = ZB = YT^{-1}TA = U_r N^{-1/2}T^{-1}TN^{1/2}\Gamma_r W_r = U_r\Gamma_r W_r \,. \tag{16}$$

The matrix Z contains scaled stimulus-pair projections on the rotated axes. Each column of Z thus provides a set of measures representing distances between pairs of stimuli in terms of a rotated dimension of viewpoint about stimulus similarity. The $n(n - 1)/2$ coefficients in each of the r columns of Z constitute measures of distance between the $n(n - 1)/2$ possible pairs of n stimuli, which may then be arrayed in r separate n by n distance matrices. Each distance matrix is then analyzed by the methods of multidimensional scaling to obtain r separate multidimensional spaces (Messick and Abelson [29], Torgerson [45], Shepard [40]).

The matrix B contains, as row vectors, scaled individual projections on the rotated axes. *The coefficients in the r rows of B may be considered scores for the individuals on r viewpoint variables.* The size of each coefficient in a row indicates the extent to which that individual's point of view about stimulus similarity corresponds to the particular rotated dimension of viewpoint represented by the row. Since each individual receives a score on all r viewpoint dimensions, correlations may be computed between these viewpoint variables and scores on other outside measures—perhaps of personality, cognitive, or social variables—to ascertain properties and correlates of the viewpoint dimensions. Scores for the individuals on outside measures may also be used in a kind of multiple-correlation procedure to orient viewpoint dimensions in the factor space of individuals (matrix B) so that they correlate as highly as possible with particular outside measures (Mosier [33], Cliff, [6]). In multiple-correlation terms, if the projection of an outside measure into the individual factor space of B is found to account for most of the measure's variance (high multiple correlation), then a viewpoint can be located (using B weights as direction numbers) and the attendant multidimensional space derived to represent high scorers on the outside measure, whether they were actually present in the sample or not.

Idealized Individuals

Since the entries in matrix B represent coordinates of points for individuals on rotated axes, this space may be readily plotted graphically. The factor space of individuals would also usually be plotted prior to rotation from the entries in matrix A. If certain individuals are of particular interest, perhaps because of their scores on other variables or because of their deviant or central location in the factor space, it may be desirable to derive separate multidimensional spaces for each of these persons. This may be accomplished by estimating distance measures $\hat{x}_{(jk)i}$ for each of these i individuals by postmultiplying matrix Z by those column vectors of B corresponding to the selected individuals. If the selected column vectors of B are referred to as B_i, then from (16)

$$\hat{X}_i = ZB_i \,, \tag{17}$$

where $\hat{X}_i$ is an $n(n - 1)/2$ by i matrix of estimated distance measures for i

selected individuals, and B_i is an r by i matrix of selected individual coefficients on rotated viewpoint dimensions. The i sets of distance measures in $\hat{X}_i$ are estimated only from the factor variance in the r-dimensional viewpoint space. Since much of the error variance in the original $x_{(jk)i}$ measures has thereby been eliminated, the reproduced distance measures $\hat{x}_{(jk)i}$ should be more stable than the raw ratings for subsequent analysis. Each of the i columns of $\hat{X}_i$, then, contains $n(n - 1)/2$ measures of distance between pairs of stimuli. These can then be analyzed separately by multidimensional scaling methods to produce i separate spaces, one for each of the selected individuals.

It is also possible to insert onto the plots of the factor space of individuals additional points at any desired location. These points may be interpreted as "idealized individuals." Their location may be determined from any desired criterion, such as placing an idealized individual near or within clusterings of points for real individuals, or at the extremities of the array of real points, or at positions determined by outside measures. Any desired number of idealized individuals may be inserted into the factor space.

Separate multidimensional spaces may be derived for each idealized individual as follows. First, read the coordinates of each idealized point from the factor plots of matrix B, and record the r coordinates of each point in a column vector. Assemble these column vectors for g idealized individuals into a matrix G. Analogously to (16) and (17), compute

$$\hat{X}_g = ZG, \tag{18}$$

where $\hat{X}_g$ is an $n(n - 1)/2$ by g matrix of estimated distance measures for g idealized individuals, and G is an r by G matrix of idealized individual coordinates on the rotated axes. The elements in each column of $\hat{X}_g$ represent estimates of distance among the possible pairs of n stimuli for each idealized individual. These distances can then be analyzed separately by multidimensional scaling methods to produce g separate spaces, one for each idealized individual.

If the idealized individual points are inserted into the factor plots prior to rotation, then the coordinates would be read from the reference frame of matrix A, and

$$\hat{X}_g = YG_A\ , \tag{19}$$

where G_A is an r by g matrix of idealized individual coordinates on the unrotated factors of matrix A.

The extent to which each real individual's point of view about stimulus similarity is related to each of r selected idealized viewpoints may be determined by rotating the dimensions of the factor space of individuals to positions defined by idealized individuals. That is, a dimension is located for each selected idealized individual on which that idealized individual has a loading of unity and the other idealized individuals have loadings of zero.

Then the projections of the real individuals on each dimension will indicate the extent of relation between the real individuals and the selected idealized viewpoint. For this purpose, G can be considered to be an extension of the B matrix. G_r can be defined as an r by r square section of G. Thus, an r by r nonsingular transformation matrix Λ is sought that will rotate the idealized individual vectors of G_r into new positions such that each transformed vector has one unit loading with the remaining entries zero loadings:

$$\Lambda G_r = I. \tag{20}$$

Although any number of idealized individuals may be defined and their corresponding distance measures derived by (18), (20) can be solved only if G is a square matrix, possessing an inverse. The coefficients relating real individual viewpoints to idealized viewpoints can be computed in stages for various square sections of G.

$$\Lambda = G_r^{-1}, \tag{21}$$

$$H = \Lambda B = G_r^{-1}B, \tag{22}$$

where H is an r by N matrix of projections of real individuals on r selected idealized individual dimensions. The size of these coefficients indicates the extent of relationship between each real individual viewpoint and the idealized individual viewpoints.

Thus, the computation of distance measures for idealized individuals is seen to be the result of another rotation on the factor space, since

$$\hat{X}_r = ZB = ZG_rG_r^{-1}B = \hat{X}_{gr}H. \tag{23}$$

Each column of a Z matrix of rotated stimulus-pair coefficients, then, may be interpreted as measures of dissimilarity or distance between pairs of stimuli for an idealized individual. Each row of the corresponding B matrix of rotated individual coefficients relates each real individual to a particular idealized viewpoint. The resultant perceptual spaces for the idealized individuals are indicative of the variety of spaces existent for the real individuals in the sample.

If all subjects in the sample happen to have similar perceptual spaces for the selected set of stimuli, there will be only one column in matrix Z, and the perceptual space for the single idealized individual will represent the space for all real individuals in the sample. At the opposite extreme, the perceptual space for each subject might be unrelated to the spaces of every other subject. In this case, there would be as many idealized individuals as real individuals, and each subject could be considered his own idealized individual. Between these two extremes there are many possible degrees of complexity in the structure of individual differences in perceptual spaces which may be investigated experimentally.

The Perceptual Space for the Group Average

Since cross products are analyzed in the present procedure rather than intercovariances, the information contained in the means of the dissimilarity ratings or distance scores $x_{(jk)i}$ is retained in the analysis. Consequently, the first characteristic root of the cross-products matrix P is very large relative to the subsequent roots, since the corresponding first principal vector in U_r essentially recovers these mean scores. Indeed, the first characteristic root is often so large relative to the remainder that some rules-of-thumb carried over from factor analyses of covariances and correlation matrices would usually indicate the presence of only one consistent factor in P. Therefore, to avoid giving undue weight to the consistently large first root of a cross-products matrix, criteria for deciding the number of factors should include, in addition to relative variance accounted for, the search for patterns in the distribution of roots and for sudden breaks in the distribution of successive differences in roots.

Although the coefficients in the first unrotated principal vector in U_r are not precisely proportional to the average $x_{(jk)i}$ values (since the first principal component accounts for somewhat more variance than would the unweighted mean dissimilarity ratings), the loadings on this first vector will be very highly correlated with the mean $x_{(jk)i}$ values and may be interpreted as distance measures for the "average person" in the group. An average perceptual space may then be derived by treating the $n(n - 1)/2$ loadings on the first unrotated principal vector in U_r as measures of distance among the n stimuli and applying the standard procedures of multidimensional scaling analysis [29, 40, 45]. In this analysis of the average perceptual space, the first unrotated U_r vector may be scaled, if desired, by $N^{-\frac{1}{2}}$ as in (9) or weighted by the corresponding latent root, since the distances and the associated perceptual space are determined only to within multiplication by positive constants [45].

Thus, a perceptual space obtained by treating the coefficients on the first principal vector in U_r as distance measures would be roughly equivalent to a multidimensional scaling of the average $x_{(jk)i}$ values. However, only in the case where a single viewpoint dimension is found to be necessary in the principal-components analysis of P would these average distance values adequately represent the individual spaces.

An Illustrative Analysis of Political Judgment Data

An analysis of judgments of dissimilarity among certain political leaders with respect to their political thinking was performed according to the present model to illustrate the procedure. Data were selected from a larger set previously analyzed by Messick [28] by traditional multidimensional scaling methods. Messick [28] asked 574 male and 262 female undergraduates to rate on a nine-point scale the similarity of all possible pairs of 20 political leaders with respect to their political thinking. The multidimensional method of successive intervals (Messick [26], Diederich, Messick, and Tucker [8]) was then applied to two separate subsamples of 267 students who endorsed the Democratic Party and 464 subjects who aligned themselves with the Republicans. The two resulting perceptual spaces each consisted of seven dimensions with essentially identical arrangements of stimulus points.

For the present purpose, a smaller sample of 39 students was selected in terms of their answers to the four questionnaire items listed in Table 1. To illustrate the advantages of the present method, an attempt was made to insure consistent individual variation in judgments of similarity by including in the analysis four groups of individuals representing four different patterns of response to those items: liberal Democrats in favor of labor, conservative Democrats in favor of management, liberal Republicans in favor of labor, and conservative Republicans in favor of management. Ten subjects were selected for each of these groups except the conservative Democrats in favor of management; it was not possible to find ten students with this latter combination of responses. Even when the selection criterion was relaxed to include conservative Democrats who did not indicate a sym-

TABLE 1

Selection of Individuals for Political Judgment Study

	Responses of Selected Individuals			
Question	Liberal Democrat	Conservative Democrat	Liberal Republican	Conservative Republican
In general, which political party do you support in most political matters?	Democratic	Democratic	Republican	Republican
On the whole, which party do you think best represents the interests of the American people?	Democratic	Democratic	Republican	Republican
How would you classify your own political position as to "liberalism" or "conservatism"?	Liberal	Conservative	Liberal	Conservative
In strikes and other disputes between management and labor, where do your sympathies usually lie?	With labor	With management Both Neither Not sure	With labor	With management

TABLE 2

Cross-Products Matrix P

Individual		1	2	3	4	5	6	7	8	9	10
	1	4894									
	2	5322	7786								
	3	5440	7451	8210							
Liberal	4	5154	6689	6989	6962						
Democrats	5	5373	7088	7395	6773	7801					
for Labor	6	5485	7484	7537	6770	7215	8155				
	7	5604	7203	7648	7174	7539	7325	8499			
	8	5631	7506	7692	7144	7596	7558	7901	8287		
	9	4937	6730	7006	6369	6723	6832	7094	7079	6999	
	10	5397	7161	7400	6596	7130	7388	7224	7415	6733	7758
	11	4565	6409	6575	5735	6151	6548	6378	6541	6099	6319
	12	4775	6488	6788	5903	6394	6597	6493	6603	5971	6501
	13	5618	7119	7237	6808	7171	7513	7435	7567	6655	7379
Conservative	14	5469	7444	7565	6801	7303	7655	7374	7607	6795	7348
Democrats	15	6294	8443	8474	7786	8213	8573	8519	8793	7662	8384
Non-Labor	16	5474	7584	7931	7256	7458	7785	7793	7830	7301	7560
	17	5714	7868	8053	7265	7707	7914	7961	8131	7393	7772
	18	5567	7330	7598	6942	7389	7641	7625	7870	6794	7325
	19	4699	6432	6628	5977	6319	6462	6562	6628	6085	6391
	20	5476	7510	7750	6747	7268	7615	7432	7645	6963	7388
	21	4758	6080	6368	5872	6189	6227	6500	6537	5645	6360
	22	4377	5982	6138	5452	5803	6194	5836	6034	5592	6106
Liberal	23	3966	5378	5484	4820	5228	5526	5300	5477	5038	5411
Republicans	24	4660	6349	6572	5777	6200	6521	6228	6461	5853	6295
for Labor	25	5024	6450	6850	6172	6527	6908	6772	6828	6350	6771
	26	4688	6439	6651	5895	6274	6573	6403	6541	5946	6398
	27	5043	6753	7026	6486	6749	6807	7064	7042	6345	6851
	28	4301	5901	5966	5318	5721	6160	5803	5955	5543	5979
	29	4099	5329	5501	5062	5333	5598	5468	5578	5084	5437
	30	4695	6628	6654	5655	6332	6679	6255	6632	5934	6431
	31	5163	7003	7347	6720	7073	7138	7552	7433	7082	7064
	32	5655	7744	7913	7161	7555	7911	7739	7884	7182	7587
Conservative	33	5727	7678	7878	7129	7719	7890	7869	7944	7214	7648
Republicans	34	6107	7870	8123	7542	7849	8264	8181	8246	7408	8190
for	35	5044	6561	6758	6213	6571	6823	6811	6859	6163	6695
Management	36	5098	7037	7207	6489	6925	7205	7114	7304	6611	6943
	37	5137	7185	7143	6445	6812	7373	6949	7337	6514	7047
	38	4729	6459	6617	5866	6130	6616	6310	6562	5896	6461
	39	4639	6267	6454	5896	6267	6364	6460	6507	5918	6445

TABLE 2 (Cont'd.)

Cross-Products Matrix P

Individual		11	12	13	14	15	16	17	18	19	20
	1										
	2										
	3										
	4										
	5										
	6										
	7										
	8										
	9										
	10										
	11	6299									
	12	5821	6388								
	13	6264	6451	8215							
Conservative	14	6545	6720	7339	7999						
Democrats	15	7175	7368	8519	8434	10544					
Non-Labor	16	6727	6765	7426	7790	8682	8905				
	17	6924	7017	7712	7951	8925	8200	8820			
	18	6477	6494	7514	7611	8727	7736	7886	8497		
	19	5721	5867	6333	6537	7270	6750	6993	6446	6142	
	20	6724	6860	7382	7699	8535	7762	8103	7550	6630	8261
	21	5408	5673	6555	6353	7417	6371	6586	6473	5580	6473
	22	5371	5486	6024	6140	6884	6256	6349	6051	5338	6287
Liberal	23	4760	4923	5312	5515	6033	5498	5747	5354	4811	5535
Republicans	24	5608	5852	6430	6517	7254	6514	6842	6495	5717	6658
for Labor	25	5833	6002	6903	6750	7728	7032	7170	6838	5875	6974
	26	5654	5918	6443	6612	7262	6651	6901	6497	5729	6734
	27	5858	6043	7002	6797	7830	7196	7313	6873	6065	6924
	28	5255	5279	5867	5994	6740	6128	6399	5937	5258	6095
	29	4739	4855	5481	5590	6332	5549	5868	5499	4848	5640
	30	5800	5970	6261	6687	7405	6600	7107	6485	5757	6840
	31	6324	6313	7100	7169	8129	7628	7781	7177	6366	7309
	32	6752	6787	7669	7859	8913	8103	8363	7819	6717	7900
Conservative	33	6814	7025	7779	7765	8912	8075	8230	7826	6829	7996
Republicans	34	7004	7131	8458	8133	9415	8435	8592	8312	7057	8266
for	35	5674	5874	6796	6758	7742	7033	7102	6787	5861	6812
Management	36	6240	6315	7102	7114	8181	7413	7628	7228	6227	7253
	37	6239	6227	6997	7251	8228	7471	7626	7249	6131	7204
	38	5713	5839	6308	6567	7338	6646	6973	6507	5775	6734
	39	5471	5654	6524	6349	7414	6723	6754	6440	5621	6468

TABLE 2 (Cont'd.)

Cross-Products Matrix P

Individual		21	22	23	24	25	26	27	28	29	30
	1										
	2										
	3										
	4										
	5										
	6										
	7										
	8										
	9										
	10										
	11										
	12										
	13										
	14										
	15										
	16										
	17										
	18										
	19										
	20										
	21	6874									
	22	5260	5763								
Liberal	23	4579	4549	4559							
Republicans	24	5528	5311	4878	6735						
for Labor	25	5937	5744	4937	5907	7425					
	26	5612	5358	4850	5914	5985	6257				
	27	6047	5530	4891	5844	6412	6024	7273			
	28	5056	5017	4495	5386	5522	5289	5338	5567		
	29	4810	4623	4041	4944	5230	4822	5045	4600	5165	
	30	5424	5372	4942	5972	5854	5850	5833	5440	4899	6645
	31	6131	5806	5225	6164	6775	6302	6876	5830	5416	6210
	32	6545	6339	5656	6985	7210	7052	7212	6212	5827	6918
Conservative	33	6659	6385	5701	6836	7182	6820	7297	6331	5729	6882
Republicans	34	7158	6756	5828	6999	7831	7032	7718	6508	6166	6950
for	35	5824	5602	4934	5790	6276	5821	6228	5454	5114	5845
Management	36	6072	5745	5152	6313	6618	6315	6730	5752	5219	6300
	37	5994	5898	5296	6165	6449	6251	6330	5838	5398	6484
	38	5612	5370	4812	5852	5914	5869	5959	5385	4912	5987
	39	5820	5246	4633	5480	5867	5571	6214	5136	4707	5524

TABLE 2 (Cont'd.)

Cross-Products Matrix P

Individual		31	32	33	34	35	36	37	38	39
	1									
	2									
	3									
	4									
	5									
	6									
	7									
	8									
	9									
	10									
	11									
	12									
	13									
	14									
	15									
	16									
	17									
	18									
	19									
	20									
	21									
	22									
	23									
	24									
	25									
	26									
	27									
	28									
	29									
	30									
	31	7691								
	32	7592	9510							
Conservative	33	7640	8085	9067						
Republicans	34	7922	8553	8537	9809					
for	35	6511	7044	7006	7577	6903				
Management	36	6990	7671	7624	7846	6392	7482			
	37	6787	7643	7390	7801	6542	6788	7708		
	38	6260	6965	6817	7062	5867	6194	6380	6379	
	39	6377	6724	6815	7276	5925	6203	6119	5624	6373

pathy toward labor, only nine such subjects could be found from the larger sample of over 800.

Dimensions of Viewpoint

The 39 individual ratings of dissimilarity for the 190 possible pairs of 20 political leaders were arrayed in the matrix X, which in this case had 190 rows for the stimulus-pairs and 39 columns for the individual raters. Each cell entry $x_{(jk)i}$ was an integer from 1 to 9, representing the category of dissimilarity into which individual i placed stimulus-pair (jk). The 20 political leaders used as stimuli are listed in Table 8.

The matrix P of sums of cross products between columns of X was computed by (1) and is presented in Table 2. P was next analyzed by the method of principal components as in (3). The diagonal matrix of characteristic roots Γ^2 contained one very large root as expected (258,784.74), then two smaller roots (3381.42 and 2524.55) that appeared in terms of the total pattern to be somewhat larger than the subsequent roots, which trailed off in fairly regular steps to near zero as follows.

1908.95	877.35	578.95	370.90	217.90
1734.56	865.00	540.14	333.48	200.29
1568.78	794.64	519.69	326.63	180.94
1407.16	758.09	486.41	307.15	140.96
1271.57	695.86	453.25	291.44	
1110.60	680.20	439.73	283.05	
1085.22	653.22	422.72	252.96	
948.28	589.55	389.77	230.03	

Consequently, it was decided to characterize the structure of individual differences in terms of three dimensions. The square roots of the three largest characteristic roots of P were used to construct the diagonal matrix Γ_r (having diagonal elements 508.71, 58.15, and 50.24), and the three corresponding characteristic vectors of P comprised the matrix W_r (Table 3).

At this point, W_r would ordinarily be rescaled to form the matrix V by multiplying each element by $\sqrt{39}$ as in (8). Since only one sample was involved in the present example, this step was left out of the computations for the sake of simplicity. Each row of W_r was weighted by the corresponding Γ_r value to produce the corresponding row of matrix A, as in (12) (see Table 3). The entries in the first row of the A matrix (or the first column of A' as in Table 3) were fairly uniform large positive values for all the individuals. As expected, however, these first factor loadings were very highly correlated

TABLE 3

Individual Coefficients on Principal Vectors W_r' and on Principal Factors A'

Individual		Matrix W_r' (or V')			Matrix A'		
		I	II	III	I	II	III
	1	.1242	-.1433	-.1266	63.18	-8.33	-6.36
	2	.1668	.1374	.1205	84.87	7.99	6.05
	3	.1717	.0302	.1968	87.36	1.75	9.89
Liberal	4	.1555	-.2426	.1588	79.09	-14.11	7.98
Democrats	5	.1651	-.1374	.1597	83.97	-7.99	8.03
for Labor	6	.1710	.1215	-.0676	86.97	7.06	-3.40
	7	.1702	-.3355	.2658	86.58	-19.51	13.36
	8	.1731	-.1371	.1479	88.08	-7.97	7.43
	9	.1565	-.0462	.2890	79.60	-2.68	14.52
	10	.1671	-.0084	-.1299	85.00	-0.49	-6.52
	11	.1471	.1614	.1288	74.82	9.38	6.47
	12	.1501	.1416	.0017	76.35	8.23	0.09
	13	.1684	-.1985	-.2975	85.66	-11.55	-14.95
Conservative	14	.1704	.1131	.0228	86.67	6.57	1.15
Democrats	15	.1933	-.1159	-.1364	98.33	-6.74	-6.85
Non-Labor	16	.1756	-.0599	.2796	89.32	-3.48	14.05
	17	.1802	.0659	.1693	91.67	3.83	8.51
	18	.1711	-.1061	-.0210	87.04	-6.17	-1.06
	19	.1482	.0433	.0972	75.41	2.52	4.89
	20	.1725	.1553	.0004	87.76	9.03	0.02
	21	.1457	-.2091	-.3643	74.10	-12.16	-18.30
	22	.1388	.1220	-.1732	70.61	7.10	-8.70
Liberal	23	.1236	.1694	-.0209	62.90	9.85	-1.05
Republicans	24	.1478	.2638	-.1204	75.16	15.34	-6.05
for Labor	25	.1560	-.0830	-.2341	79.35	-4.82	-11.76
	26	.1487	.1526	-.0222	75.64	8.87	-1.11
	27	.1574	-.2348	.0249	80.05	-13.65	1.25
	28	.1365	.1886	-.0794	69.45	10.97	-3.99
	29	.1264	.0538	-.1852	64.31	3.13	-9.30
	30	.1493	.3538	.0160	75.93	20.57	0.81
	31	.1651	-.1267	.2186	83.99	-7.37	10.99
	32	.1786	.1194	.0446	90.85	6.94	2.24
Conservative	33	.1784	-.0096	.0396	90.76	-0.56	1.99
Republicans	34	.1866	-.1859	-.2828	94.94	-10.81	-14.21
for	35	.1540	-.0836	-.1332	78.33	-4.86	-6.69
Management	36	.1631	.0117	.0479	82.95	0.68	2.41
	37	.1629	.1739	-.0307	82.85	10.11	-1.54
	38	.1487	.2094	-.0617	75.65	12.18	-3.10
	39	.1470	-.1573	-.1048	74.77	-9.15	-5.26

(Spearman rank r of .97) with the average ratings made by these individuals to all 190 stimulus-pairs.

The matrix U_r of stimulus-pair projections on the unrotated principal vectors was computed by (4) (Table 4). Since the scaling factor for sample size was not included in the present example, this step also corresponds to the computation of the matrix Y by (11).

Variation in Individual Additive Constants

The present model assumes that the input data or $x_{(jk)i}$ values represent estimates of distance between stimuli j and k for each individual i. Thus, in the case of one average dimension of viewpoint the elements of the matrices in (16) could be represented by

$$\hat{x}_{(jk)i} = z_{(jk)}b_i \ , \tag{24}$$

where $z_{(jk)}$ is a loading for the (jk)th stimulus-pair on the single dimension, and b_i is a weight for the ith individual. For r dimensions of viewpoint,

$$\hat{x}_{(jk)i} = \sum_{m=1}^{r} z_{(jk)m}b_{mi} \ , \tag{25}$$

where $z_{(jk)m}$ is a loading for the stimulus-pair on the mth dimension of viewpoint, and b_{mi} is a weight for the individual on the mth dimension.

In terms of the model, these distances should be measured on a ratio scale and if they are not, certain variations in individual scale properties might be mistaken for individual differences in viewpoint. For example, even though interval properties may be reflected in a distance scale based upon category ratings of stimulus dissimilarity, such as the procedure used in the present study, the zero points of the scales for individuals might not be comparable. Thus, for the case of one underlying viewpoint dimension, (24) would become

$$\hat{x}_{(jk)i} = z_{(jk)}b_i + c_i \ , \tag{26}$$

where c_i is an additive constant to translate each individual's scale to a ratio scale with a fixed zero point (cf. Messick and Abelson [29]). For r dimensions of viewpoint,

$$\hat{x}_{(jk)i} = \sum_{m=1}^{r} z_{(jk)m}b_{mi} + c_i \ . \tag{27}$$

The right side of (26) can be represented in matrix terms as a Z matrix (comprised of a column vector of $z_{(jk)}$ loadings on the single viewpoint dimension and a column vector of unities) times a B matrix (composed of a row vector of b_i weights and a row vector of individual additive constants c_i). The X matrix of (26) is thus seen to be of rank 2 even though only one underlying viewpoint dimension was postulated. Similarly, the right side of (27)

TABLE 4

Stimulus-Pair Projections on Principal Vectors Matrix U_r (or Y)

Stimulus-Pairs	I	II	III	Stimulus-Pairs	I	II	III
1	.0736	.0207	.0172	41	.0493	.0080	-.0931
2	.0485	.0496	-.1295	42	.0906	.0527	.0484
3	.0765	-.1392	-.0468	43	.0720	-.0672	.1348
4	.0664	.0374	-.0234	44	.0797	-.1286	.0656
5	.0752	.0249	.0366	45	.0714	.0036	.0156
6	.0942	.0659	-.0490	46	.0603	-.0630	.0288
7	.0691	.0088	-.1396	47	.0623	-.1077	.0326
8	.0857	.1054	-.0373	48	.0680	-.0752	-.0901
9	.0428	.0369	-.0086	49	.0800	-.1174	.0783
10	.0870	-.0170	-.0160	50	.0715	-.0182	.0359
11	.0779	-.1356	.0793	51	.0610	-.0846	.0481
12	.0573	.0893	-.0943	52	.0516	.0062	-.0974
13	.0564	.0454	.0507	53	.0701	-.0065	.0694
14	.0750	-.0240	-.0402	54	.0995	.0482	.0361
15	.0656	.0392	.0633	55	.1000	.0785	.0295
16	.0659	-.0605	.0646	56	.0856	.2322	.0616
17	.0502	.0482	-.0570	57	.0790	.1291	.0998
18	.0723	-.0465	-.0184	58	.0770	-.0638	-.0193
19	.0894	.0472	.0273	59	.1017	.0667	.0131
20	.0783	.0414	.0386	60	.0896	.0793	.0560
21	.0671	.0165	-.1056	61	.0756	.1862	.0502
22	.0946	.0571	-.0033	62	.0994	.0701	.0624
23	.0548	.0888	-.0269	63	.0433	.0116	-.1634
24	.0673	.0291	-.0617	64	.0574	-.1003	.0404
25	.0675	-.0399	.0083	65	.0372	.0089	-.0459
26	.0599	.0774	-.0269	66	.0962	.0968	.0243
27	.0459	.0056	-.0926	67	.0857	-.0658	-.0756
28	.0699	-.0550	-.0111	68	.0707	-.0923	-.0360
29	.0936	.0389	.0562	69	.0713	-.0029	-.0321
30	.0708	-.1067	-.0656	70	.0740	.0202	-.0080
31	.0982	.0116	.0086	71	.0758	-.0358	-.0884
32	.0618	-.0201	-.0306	72	.0380	.0673	-.1082
33	.0648	-.1465	.0582	73	.0515	.0656	-.0947
34	.0822	.0031	-.0083	74	.0786	-.0249	-.0097
35	.0533	.0135	.0105	75	.0687	-.0866	.0939
36	.0646	-.1282	.0076	76	.0432	-.0262	-.0356
37	.0693	-.0139	-.0629	77	.0675	-.1301	.0159
38	.0550	.0166	.0272	78	.0650	-.0488	-.0272
39	.0921	.0038	.0446	79	.0562	.0321	.0579
40	.0488	.0467	-.1238	80	.0658	.0124	-.0977

TABLE 4 (Cont'd.)

Stimulus-Pair Projections on Principal Vectors Matrix U_r (or Y)

Stimulus-Pairs	I	II	III	Stimulus-Pairs	I	I	III
81	.0888	-.0062	-.0026	121	.0488	-.0351	-.1233
82	.0974	.0867	.0355	122	.0976	.0924	.0129
83	.0409	.0112	-.1776	123	.0822	.0540	.1195
84	.0934	.0438	.0227	124	.0568	-.0644	-.1068
85	.0861	.0074	.0204	125	.0750	-.0295	.0299
86	.0710	-.0141	-.0124	126	.0742	-.1109	-.0367
87	.0722	-.1096	-.0575	127	.0307	-.0334	-.0440
88	.0645	-.0818	.0445	128	.0716	-.1162	.0819
89	.0787	.0046	.0132	129	.0675	-.0484	-.0219
90	.0491	-.0792	-.1036	130	.0566	-.0435	-.0800
91	.0622	.1536	.0632	131	.1021	.0818	.0467
92	.0844	.0254	.0076	132	.0846	.1381	.0224
93	.1021	.0574	.0400	133	.0668	-.0704	-.0336
94	.1029	.0743	-.0107	134	.0721	-.0933	.0775
95	.0734	-.0795	.0565	135	.0685	-.0071	-.0622
96	.0730	-.0844	.0428	136	.0647	.0268	-.0742
97	.0659	-.0892	.0347	137	.0675	-.0612	.0017
98	.0647	-.1298	.0163	138	.0914	.0575	.0680
99	.0901	.0394	-.0301	139	.0659	-.1005	.0910
100	.0673	-.0491	-.0172	140	.0803	-.0616	.0362
101	.1037	.0919	-.0039	141	.0589	.0238	-.0698
102	.0994	.0662	.0503	142	.0345	-.0195	-.1535
103	.1008	.1038	-.0222	143	.0666	-.1292	.0164
104	.0759	.0057	-.0242	144	.0649	-.0026	.0285
105	.0614	.0058	-.0872	145	.0571	-.0330	-.1518
106	.0890	.0767	.1453	146	.0961	.0382	.0700
107	.0420	.0072	-.0549	147	.0671	-.1552	.0936
108	.0674	-.0346	.0972	148	.1022	.0585	-.0033
109	.0506	.0131	-.1498	149	.0673	-.0923	.0156
110	.0997	.0668	.0252	150	.0943	.0504	.0439
111	.0633	-.1446	.0305	151	.0669	-.0661	.0482
112	.0728	-.0188	-.0795	152	.0569	-.0304	-.1791
113	.0760	-.0570	.0248	153	.0531	.0351	-.1875
114	.0769	-.0692	.0652	154	.0652	.0142	-.0940
115	.0740	-.0915	.0555	155	.0636	-.0412	-.0926
116	.0371	-.0188	-.1751	156	.0821	.2234	.0518
117	.0719	-.0850	-.0256	157	.0706	-.0743	.0946
118	.0951	.0355	.0564	158	.0706	-.0367	.0076
119	.0966	.0690	.0147	159	.0624	-.0790	.0516
120	.0939	.0993	.1064	160	.0747	-.0334	.0368

TABLE 4 (Cont'd.)

Stimulus-Pair Projections on Principal Vectors Matrix U_r (or Y)

Stimulus-Pairs	I	II	III	Stimulus-Pairs	I	II	III
161	.0504	.0237	-.1404	181	.0966	.0497	.0131
162	.0996	.0787	.0871	182	.0776	-.0326	.1007
163	.0457	.0383	-.1582	183	.0695	-.0601	-.0393
164	.0464	.0134	-.1650	184	.0526	.0098	-.0306
165	.0524	-.0230	-.0021	185	.0998	.1019	.0408
166	.0433	-.0185	-.1012	186	.0399	-.0113	-.1420
167	.0582	-.0613	.0604	187	.0601	-.0176	-.0569
168	.0678	-.0775	.0173	188	.0561	-.0227	-.1357
169	.1020	.0580	-.0063	189	.0398	.0306	-.1480
170	.0436	-.0253	-.1663	190	.0596	.0062	.0203
171	.0625	-.0416	-.0321				
172	.0873	.1479	.0402				
173	.0699	-.0785	-.0075				
174	.0677	-.0436	-.0149				
175	.0288	-.0196	-.0737				
176	.0542	-.1371	.0526				
177	.0570	-.1135	.0332				
178	.0716	.0463	.0750				
179	.0464	-.0012	-.0552				
180	.0744	-.1009	.0120				

can be represented as a Z matrix with r columns of $z_{(jk)}$ values and a column of unities times a B matrix with r rows of b_i weights and a row of c_i values. The corresponding X matrix is thus seen to be of rank $r + 1$ even though only r viewpoint dimensions were postulated.

Thus, the possibility that one of r dimensions obtained with the present procedure represents variations in individual scale constants and not a dimension of viewpoint should be carefully evaluated, particularly if interval scaling or rating procedures had been used to estimate the distances originally. This evaluation can be achieved by determining an r by r transformation matrix L that will rotate Z (or U_r) as closely as possible in a least-squares sense to a matrix Q that contains a column of unities (or, because of free transformation with a scale factor, a column of constants):

$$U_r L = \hat{Q}, \tag{28}$$

where $\hat{Q}$ is a least-squares estimate of the criterion matrix Q. If $\hat{Q}$ is found to contain a column of unities or constants, within some acceptable range of variation, then one of the r dimensions of X is interpretable in terms of individual variations in scale constants, with the remaining $r - 1$ dimensions representing different viewpoints about stimulus similarity. The inverse transformation L^{-1} applied to the appropriate matrix of individual weights will then provide the corresponding c_i values.

A least-squares solution to (28) has been outlined by Cliff [6] in which

$$L = (U_r'U_r)^{-1}U_r'Q(I - \Phi)^{-1} \tag{29}$$

and

$$(1 - \varphi_m) = \frac{Q_m'U_r(U_r'U_r)^{-1}U_r'Q_m}{Q_m'Q_m}, \tag{30}$$

where Q_m is the mth column of Q. In the present case, since $U_r'U_r = I$, these formulas simplify to

$$(1 - \varphi_1) = \frac{(Q_1'U_r)(Q_1'U_r)'}{(Q_1'Q_1)}, \tag{31}$$

where

Q_1 is the column of Q containing the unities,
$(Q_1' U_r)$ is a row vector of column sums of U_r ,
$(Q_1' Q_1)$ is the number of stimulus-pairs $= n(n - 1)/2$.

$$\hat{Q}_1 = U_rL_1 = U_r(U_r'Q_1)(1 - \varphi_1)^{-1}, \tag{32}$$

where

$\hat{Q}_1$ is a least-squares estimate of the column of unities,
L_1 is the column of L that transforms U_r into $\hat{Q}_1$,
$(U_r' Q_1) = (Q_1' U_r)'$ is a column vector of column sums of U_r .

If it is desired to maintain $\hat{Q}_1$ as a unit-length vector, then the transformation simply entails postmultiplying U_r by a column vector containing the r column sums of U_r , this column vector being normalized to unit length (i.e., scaled so that the sums of squares of the r column sums becomes unity).

The $\hat{Q}_1$ vector was computed from the U_r matrix in the present example, and instead of containing relatively constant values, its entries varied widely between .0389 and .1002. (If all of the entries were equal, each of the 190 $\hat{q}_1$ values would have been approximately .0726 in the unit-length vector form.) Since it was thus not possible to determine a direction in the obtained three-dimensional factor space that would clearly correspond to variation in individual scale constants, it was concluded that all three dimensions should be interpreted in terms of differential viewpoints about stimulus similarity.

Idealized Individual Dimensions

Graphs were plotted for the three dimensions of the factor space of individuals by taking the entries in each column of matrix A (each row of A' in Table 3) to represent the coordinates of a point in three-dimensional space,

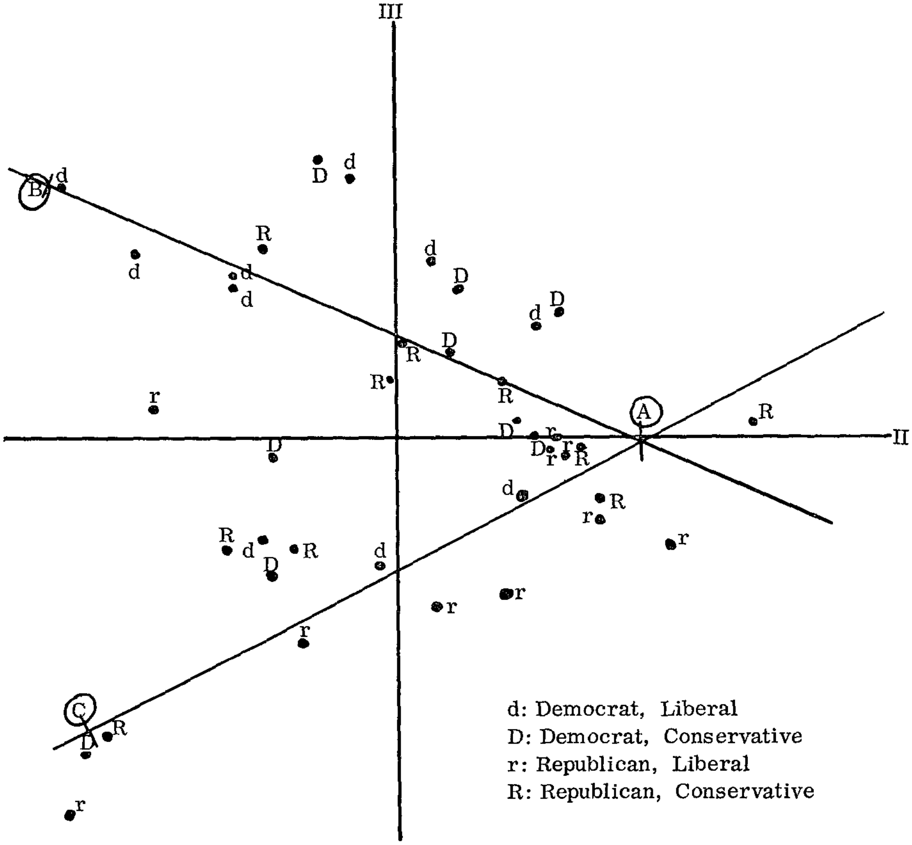

FIGURE 1

Factor Structure of Individuals,
Politics Interpoint Distance Data*

there being one point for each of the 39 individuals. The first factor in matrix A generated plots in which all individuals had large positive coordinates. The plot for the second and third factors is presented in Fig. 1.

At this point, matrix A could be rotated to simple structure as in (14) to produce matrix B of individual coefficients on r rotated axes. The inverse transformation applied to matrix Y as in (15) would then produce matrix Z of stimulus-pair projections on the rotated axes. Each column of Z would provide a set of distance measures to be analyzed by standard multidimensional scaling methods to obtain three separate perceptual spaces. Points representing idealized individuals could also be inserted into the factor

*All individuals had approximately equal large positive coefficients on the principal factor I. The graph gives the plot between coefficients on principal factors II and III.

space of matrix B, and their projections used to construct matrix G. Interstimulus distance estimates for each idealized individual would then be computed by (18), each set of distances being subsequently analyzed by multidimensional scaling methods to produce a perceptual space for each idealized individual.

In the present example, however, the factor space of matrix A (see Fig. 1) was simple enough to permit location of some idealized individual points directly, without requiring a prior rotation. An examination of Fig. 1 reveals a concentration of individuals with positive coefficients on dimension II and small or near-zero coefficients on dimension III. Many of these individuals were Republican students (as indicated by the r and R notation on the figure). The points toward the left in Fig. 1, corresponding to negative coefficients on dimension II, were more spread out on dimension III than the points on the right, giving the appearance of a triangle to the entire group of points. Two lines were drawn near the boundaries of this triangle, intersecting at the point A. Points B and C were chosen near the extreme real individual points at the left on the two lines. These three points were taken to represent three idealized individuals, chosen to span the boundaries of the real points. Their projections on dimensions II and III were obtained from Fig. 1 and used to construct the matrix G (Table 5); the projections of these idealized individuals on dimension I were taken as the mean of the coefficients for nearby real individuals on dimension I.

TABLE 5

Matrix G_A of Idealized Individual Projections on Reference Factors of Matrix A

	Idealized Individuals		
Factor	A	B	C
I	80.0	87.0	86.0
II	14.0	-20.0	-12.0
III	-1.0	13.0	-14.0

It is of interest to note the progression in Fig. 1 from liberal Republicans below the AC line through a group of conservatives above this line to a concentration of liberal Democrats toward the point B. Such an arrangement suggests that this direction may be related to individual differences on some measure of political ideology.

Since the coordinates in the matrix G were based upon the reference axes of matrix A rather than the rotated axes of matrix B, the matrix $\hat{X}_g$ of interstimulus distance estimates for the three idealized individuals was computed by (19) (Table 6). The coefficients in each of the three columns of $\hat{X}_g$ represent measures for each of the three idealized individuals of the 190

TABLE 6

Matrix $\hat{X}_g$ of Stimulus-Pair Projections on Idealized Individual Dimensions $(\hat{X}_g = Y\,G_A)$

Stimulus-Pairs	Idealized Individual Dimensions A	B	C	Stimulus-Pairs	Idealized Individual Dimensions A	B	C
1	6.158	6.210	5.839	41	4.149	2.919	5.448
2	4.707	1.546	5.392	42	7.939	7.458	6.484
3	4.218	8.832	8.905	43	4.688	9.364	5.114
4	5.858	4.724	5.588	44	4.507	10.354	7.474
5	6.328	6.522	5.656	45	5.745	6.341	5.877
6	8.508	6.239	7.996	46	3.912	6.880	5.537
7	5.792	4.023	7.792	47	3.446	8.000	6.196
8	8.370	4.864	6.628	48	4.481	6.252	8.015
9	3.953	2.877	3.361	49	4.677	10.324	7.192
10	6.741	7.704	7.914	50	5.430	7.050	5.865
11	4.253	10.518	7.214	51	3.644	7.621	5.585
12	5.927	1.973	5.176	52	4.310	3.096	5.724
13	5.095	4.654	3.593	53	5.446	7.128	5.131
14	5.705	6.483	7.302	54	8.596	8.158	7.469
15	5.731	5.744	4.284	55	9.073	7.518	7.249
16	4.361	7.783	5.489	56	10.040	3.606	3.716
17	4.751	2.667	4.540	57	8.026	5.588	3.847
18	5.149	6.979	7.030	58	5.283	7.720	7.654
19	7.789	7.191	6.742	59	9.058	7.684	7.763
20	6.807	6.488	5.699	60	8.221	6.937	5.970
21	5.703	4.133	7.048	61	8.602	3.505	3.562
22	8.374	7.049	7.501	62	8.868	8.054	6.832
23	5.653	2.641	4.024	63	3.792	1.412	5.874
24	5.856	4.475	6.307	64	3.146	7.522	5.573
25	4.833	6.778	6.166	65	3.148	2.464	3.738
26	5.904	3.316	4.601	66	9.029	6.754	6.774
27	3.839	2.675	5.174	67	6.009	7.788	9.218
28	4.833	7.038	6.827	68	4.397	7.525	7.687
29	7.978	8.100	6.799	69	5.693	5.841	6.613
30	4.238	7.443	8.289	70	6.211	5.930	6.234
31	8.010	8.422	8.185	71	5.651	6.162	8.187
32	4.693	5.380	5.985	72	4.094	0.558	3.980
33	3.077	9.327	6.518	73	5.132	1.934	4.965
34	6.628	6.980	7.148	74	5.947	7.206	7.192
35	4.443	4.506	4.276	75	4.189	8.927	5.632
36	3.369	8.287	6.991	76	3.125	3.820	4.529
37	5.410	5.488	7.006	77	3.560	8.678	7.141
38	4.606	4.806	4.151	78	4.543	6.277	6.555
39	7.373	8.514	7.248	79	4.885	4.997	3.635
40	4.683	1.702	5.371	80	5.533	4.205	6.875

TABLE 6 (Cont'd.)

Matrix $\hat{X}_g$ of Stimulus-Pair Projections on Idealized Individual Dimensions ($\hat{X}_g = Y\,G_A$)

Stimulus-Pairs	Idealized Individual Dimensions A	B	C	Stimulus-Pairs	Idealized Individual Dimensions A	B	C
81	7.019	7.817	7.749	121	3.539	3.347	6.346
82	8.971	7.202	6.839	122	9.086	6.808	7.101
83	3.605	1.023	5.867	123	7.210	7.623	4.745
84	8.063	7.546	7.191	124	3.752	4.845	7.156
85	6.968	7.603	7.027	125	5.560	7.509	6.389
86	5.493	6.297	6.447	126	4.417	8.193	8.222
87	4.298	7.724	8.328	127	2.029	2.763	3.652
88	3.970	7.825	5.905	128	4.018	9.616	6.403
89	6.349	6.929	6.532	129	4.740	6.552	6.689
90	2.919	4.506	6.620	130	3.995	4.752	6.507
91	7.064	3.163	2.623	131	9.270	7.856	7.148
92	7.101	6.936	6.849	132	8.683	4.893	5.308
93	8.927	8.251	7.529	133	4.396	6.786	7.063
94	9.283	7.330	8.109	134	4.387	9.150	6.238
95	4.702	8.709	6.475	135	5.444	5.296	6.850
96	4.615	8.595	6.690	136	5.625	4.128	6.282
97	3.988	7.968	6.252	137	4.538	7.116	6.512
98	3.341	8.433	6.891	138	8.049	7.684	6.217
99	7.792	6.661	7.699	139	3.773	8.925	5.599
100	4.715	6.613	6.619	140	5.525	8.690	7.139
101	9.582	7.179	7.820	141	5.117	3.744	5.761
102	8.828	7.975	7.049	142	2.639	1.396	5.350
103	9.538	6.404	7.732	143	3.505	8.595	7.051
104	6.176	6.175	6.798	144	5.126	6.068	5.212
105	5.083	4.096	6.436	145	4.258	3.655	7.432
106	8.045	8.095	4.696	146	8.152	8.505	6.823
107	3.519	2.800	4.298	147	3.104	10.160	6.324
108	4.814	7.822	4.852	148	8.995	7.674	8.129
109	4.379	2.192	6.290	149	4.076	7.905	6.677
110	8.886	7.666	7.420	150	8.209	7.771	6.894
111	3.008	8.793	6.750	151	4.382	7.771	5.875
112	5.640	5.674	7.598	152	4.306	3.230	7.766
113	5.259	8.076	6.874	153	4.924	1.475	6.767
114	5.115	8.919	6.528	154	5.511	4.167	6.755
115	4.585	8.990	6.687	155	4.603	5.152	7.258
116	2.881	1.328	5.869	156	9.645	3.350	3.657
117	4.588	7.623	7.563	157	4.515	8.860	5.641
118	8.046	8.295	6.961	158	5.129	6.979	6.409
119	8.682	7.218	7.277	159	3.838	7.683	5.595
120	8.797	7.569	5.396	160	5.468	7.642	6.306

TABLE 6 (Cont'd.)

Matrix $\hat{X}_g$ of Stimulus-Pair Projections on Idealized Individual Dimensions ($\hat{X}_g = Y\,G_A$)

Stimulus-Pairs	Idealized Individual Dimensions A	B	C	Stimulus-Pairs	Idealized Individual Dimensions A	B	C
161	4.507	2.089	6.019	181	8.414	7.584	7.532
162	8.982	8.222	6.400	182	5.650	8.712	5.654
163	4.353	1.155	5.688	183	4.762	6.741	7.252
164	4.068	1.626	6.144	184	4.377	3.984	4.837
165	3.873	4.994	4.812	185	9.368	7.174	6.787
166	3.302	2.819	5.360	186	3.177	1.854	5.558
167	3.735	7.070	4.892	187	4.620	4.844	6.178
168	4.320	7.673	6.518	188	4.305	3.571	6.996
169	8.981	7.635	8.166	189	3.757	0.924	5.126
170	3.297	2.134	6.378	190	4.833	5.324	4.766
171	4.448	5.851	6.322				
172	9.013	5.160	5.170				
173	4.500	7.554	7.058				
174	4.823	6.573	6.559				
175	2.108	1.943	3.747				
176	2.365	8.143	5.571				
177	2.937	7.659	5.797				
178	6.299	6.274	4.549				
179	3.755	3.346	4.782				
180	4.523	8.643	7.437				

interpoint distances spanning the 20 stimulus points. The relation of each real individual to these three idealized viewpoints can now be computed by (22).

Multidimensional Perceptual Spaces

The distance measures in each column of $\hat{X}_g$ were sorted into the appropriate order and arrayed in a separate distance matrix D. These three distance matrices were then separately analyzed by the multidimensional scaling procedures outlined by Messick and Abelson [29] and by Torgerson [45]. Even though insufficient variation was found above in individual additive constants to generate a dimension in the factor space of individuals, each of the present distance matrices might still require the determination of an additive constant, presumably of roughly comparable size for the three matrices, for an optimal dimensional resolution. In the present example, however, after one cycle of the iterative solution with an initial constant of zero (Messick and Abelson [29]), the additive constants were judged to be negligible for these three distance matrices. (Incidentally, the size of the distance estimates and the presence or absence of negative distances can be manipulated by moving the location of the idealized individual point in the factor space, particularly in relation to the large "average" first factor. Thus, it is possible to recast the additive constant problem as a problem of rotational placement of the idealized individual dimension with respect to the large average dimension.)

Three scalar-products matrices, computed from the distances in each of the three D matrices, were then analyzed separately by the method of characteristic roots and vectors. The three sets of characteristic roots are given in Table 7. An examination of these roots suggests that the perceptual space for idealized individual A is strongly unidimensional, that the space for idealized individual B has two large dimensions followed by a possible third small dimension, and that the space for idealized individual C is somewhat more complex, involving possibly five or six dimensions.

TABLE 7

Characteristic Roots of Scalar Products Matrices for Distance Measures Obtained from Three Idealized Viewpoints

	Idealized Viewpoint		
Characteristic Root	A	B	C
1	157.018	221.035	117.456
2	37.713	109.349	62.848
3	32.057	47.996	51.192
4	26.210	32.437	42.126
5	22.451	27.973	38.397
6	19.465	20.218	34.966
7	17.161	16.258	26.722
8	13.911	14.148	19.424
9	13.542	10.737	16.229
10	10.775	10.221	13.231
11	9.132	4.170	9.878
12	7.389	2.762	5.398
13	4.076	0.000	3.597
14	0.000	-3.446	0.673
15	-0.692	-8.131	0.000
16	-1.662	-11.611	-4.020
17	-5.758	-13.476	-7.347
18	-6.557	-17.715	-9.976
19	-11.583	-23.869	-15.303
20	-14.802	-32.159	-18.698

Stimulus projections on these large dimensions for each of the three idealized spaces are presented in Table 8. These stimulus values are determined within a rotation, translation, and multiplication by positive constants. The large single dimension in perceptual space A appears to reflect an evaluative distinction among the stimuli. One of the two large dimensions in space B contrasts Republicans and Democrats, and the other dimension appears to be evaluative in nature. The five or six dimensions of space C present no immediate clear distinctions of the relatively simple type that emerged in the other two spaces. This suggests that a more elaborate rotation of space C is required before interpreting the dimensions. An analysis of the perceptual space for the group average, as determined from the total sample, produced seven dimensions, which are described in detail by Messick [28].

In addition to the possibility mentioned above of relating variation in the factor space of individuals to personality and cognitive variables, it is also of interest to inquire about possible correlates of the shift in complexity of perceptual structures, from simple spaces for individuals on the AB line to the complex space for idealized individual C. Perhaps this dimension of individual differences contrasts persons that might be termed "abstract" with others that require considerable concrete and specific detail for their decisions (cf. Harvey, Hunt, and Schroder [14]). Perhaps it is related to individual differences in "cognitive complexity" (Bieri and Blacker [5], Scott [37]) or is a consequence of consistencies in preferred category widths or equivalence ranges (Messick and Kogan [30], Sloane, Jackson, and Gorlow [42]).

In conclusion, the present analysis illustrates the power of the proposed method to yield a multidimensional description of the perception of relations between stimuli by various individuals, in a framework that permits the varieties of consistent individual perceptions to be ascertained and related to other personality and cognitive variables.

Summary of the Procedure for Determining Dimensions of Individual Differences in Multidimensional Scaling

1. Obtain estimates of distance or dissimilarity between all possible pairs of n stimuli for each of N individuals. Array these distance estimates $x_{(jk)i}$ in a matrix X, having $n(n - 1)/2$ rows for the stimulus-pairs and N columns for the individuals.

2. Compute an N by N matrix of cross products $P = X'X$. If $N > n(n - 1)/2$, compute the cross-products matrix summing over the variables on the longer side of X. In this case, $P = XX'$, and a symmetric analysis is used in place of the following steps. See equations (1a)–(4a).

3. Factor P by the method of principal components and construct the diagonal matrix Γ_r^2 from the r largest characteristic roots of P and the matrix W_r from the corresponding characteristic vectors. $\hat{P}_r = W_r' \Gamma_r^2 W_r$.

TABLE 8

Stimulus Projections on Dimensions of Three Perceptual Spaces, One for Each Idealized Viewpoint

	Perceptual Spaces for Idealized Viewpoints								
	A	B		C					
Stimulus	I	I	II	I	II	III	IV	V	VI
1. Chiang Kai-shek	-0.599	0.239	1.530	-2.643	-0.285	2.413	0.764	-1.852	0.631
2. Thomas Dewey	1.615	3.823	0.781	-2.027	-1.597	0.756	0.550	2.841	-0.776
3. Senator E. Dirksen	1.643	3.988	1.017	0.272	-0.563	-2.278	-0.866	0.023	-0.359
4. Senator P. Douglas	1.781	-4.384	0.356	0.397	3.390	-0.303	-0.771	1.048	3.022
5. Dwight D. Eisenhower	1.966	2.405	2.524	-3.302	-0.447	-0.182	0.285	-0.382	0.652
6. Senator George of Ga.	0.811	-2.712	1.951	-0.328	1.014	-1.397	-0.021	2.064	1.813
7. Alger Hiss	-5.562	-0.441	-3.766	3.014	-1.617	0.821	1.367	0.124	1.135
8. Adolph Hitler	-6.702	1.506	-4.331	3.689	-0.824	1.174	-0.314	0.519	-1.802
9. Senator E. Kefauver	1.824	-4.060	0.868	-0.739	1.178	-2.729	-0.657	-1.247	-0.909
10. General D. MacArthur	1.759	2.852	2.344	-2.766	-1.171	0.479	-2.737	-0.319	0.027
11. Senator J. McCarthy	2.175	3.823	-2.667	2.984	-0.199	-1.585	-2.682	-0.117	-1.178
12. Jawaharlal Nehru	-0.941	-0.675	1.326	-2.265	-0.967	1.326	-0.013	-2.651	0.079
13. Richard Nixon	1.395	4.086	2.635	-2.181	-2.518	-1.744	1.880	0.863	-2.012
14. Franklin D. Roosevelt	0.184	-4.662	0.935	-0.442	3.278	3.133	-0.574	-0.545	-1.482
15. Joseph Stalin	-6.781	1.179	-4.776	4.048	-0.010	2.037	-1.023	0.517	-0.599
16. Adlai Stevenson	1.839	-4.257	0.554	-1.999	3.150	0.183	0.647	1.851	-1.513
17. Senator R. Taft	1.793	5.319	1.605	-2.782	-1.042	-0.536	-0.598	-0.059	1.132
18. Governor Talmadge of Ga.	0.379	-3.168	0.648	2.789	0.214	-1.955	0.135	-2.481	0.474
19. Harry Truman	0.885	-4.414	0.079	1.833	1.910	-0.881	4.091	-0.961	-0.272
20. Henry Wallace	0.535	-0.447	-3.613	2.448	-2.893	1.266	0.536	0.766	1.938

4. Scale W_r for differences in sample size to produce the matrix $V = N^{\frac{1}{2}} W_r$.

5. Compute the matrix $Y = U_r N^{-\frac{1}{2}}$ either by first obtaining $U_r = X W_r' \Gamma_r^{-1}$ or directly from $Y = X V' \Gamma_r^{-1} N^{-1}$.

6. Compute the factor matrix of individuals $A = \Gamma_r V = N^{\frac{1}{2}} \Gamma_r W_r$.

7. Plot the r factors of A graphically to determine (i) rotation to structure in the factor space of individuals and (ii) locations for idealized individuals.

8. Determine, either graphically or analytically (Harman [13]), an r by r nonsingular transformation matrix T to rotate the principal factors of A to a desired structure, denoted matrix $B = TA$.

9. Compute matrix $Z = YT^{-1}$.

10. Each column of Z contains scaled stimulus-pair projections on rotated axes. These entries represent distances between pairs of stimuli according to r rotated dimensions of viewpoint. Next, r distance matrices are constructed, one from each column of Z. These distance matrices are analyzed separately by standard multidimensional scaling procedures to obtain r perceptual spaces (Messick and Abelson [29]; Torgerson [45]; Shepard [40]).

11. If desired, the entries in the first unrotated principal factor of Y (or of U_r) may be similarly used to construct a distance matrix. Multidimensional scaling of this distance matrix produces a perceptual space for the group average.

12. If desired, locate points to represent g idealized individuals in the factor space of matrix B. Read the coordinates of each idealized point directly from the factor plots and record the r coordinates of each point in a column vector. Assemble these column vectors for g idealized individuals into an r by g matrix G.

13. Compute $\hat{X}_g = ZG$, an $n(n-1)/2$ by g matrix of estimated distance measures for g idealized individuals. (If the coordinates of the idealized points are determined from the unrotated axes of matrix A to form G_A, compute $\hat{X}_g = YG_A$.)

14. Construct a distance matrix from each of the g columns of $\hat{X}_g$ and analyze them separately by multidimensional scaling methods to obtain g perceptual spaces, one for each idealized individual.

15. If desired, for r by r square sections of G, compute $H = G_r^{-1}B$ to obtain a matrix H of projections of real individuals on r selected idealized individual dimensions.

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