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SOME MATHEMATICAL NOTES ON THREE-MODE FACTOR ANALYSIS*

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The model for three-mode factor analysis is discussed in terms of newer applications of mathematical processes including a type of matrix process termed the Kronecker product and the definition of combination variables. Three methods of analysis to a type of extension of principal components analysis are discussed. Methods II and III are applicable to analysis of data collected for a large sample of individuals. An extension of the model is described in which allowance is made for unique variance for each combination variable when the data are collected for a large sample of individuals.

Extension of the two-mode factor analytic model to three or more modes of data classification has been suggested by Tucker. Initial discussions of this development appear in the monographs: *Problems in Measuring Change* [8] and *Contributions to Mathematical Psychology* [9]. The latter of these two monographs gives the basic mathematical structure of the proposed model. A further discussion of the mathematical structure was given by Levin in his PhD dissertation *Three-mode factor analysis* [4]. Results of experimental trials of the method were reviewed by Tucker in a paper read at the 1964 Invitational Conference on Testing Problems [10]. Since the Tucker and Levin descriptions of the mathematical structure of the model and analysis procedures, there have been several mathematical developments which add power and clarity to the structure of the model. The structure of the three-mode factor analytic model is discussed here in terms of the newer mathematical statements. A further refinement to be considered involves allowances for a type of unique variance related to errors of measurement. A fictitious body of data is used to illustrate several points.

Remarks on Notation

In the development of the three-mode factor analysis model it has been found quite useful to adopt several rather unique features of notation. Some of these notational items are at variance with common mathematical usage but have been found helpful in consideration of some relatively complex relations. Much of standard summational and matrix notation has been

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retained. Following is a summary of notational items relevant to the present statement of the model.

A first item is the use of the word *mode*. Tucker introduced this term to denote "a set of indices by which data might be classified" ([9], p. 112). For example, the scores of a sample of individuals on a battery of tests could be classified by the individuals in the sample and cross-classified by the tests in the battery. The individuals in the sample would be the elements of one set of indices by which the scores are classified; thus, the sample of individuals would constitute one mode of the data. A second mode of this data would be the battery of tests. The test scores could be arranged in a rectangular table with rows for individuals and columns for tests. Such an arrangement will be termed, in the present context, a two-mode matrix. If the battery of tests were administered to the sample of individuals on several occasions, the set of occasions would be considered as a third mode. The data, now, could be arranged in a rectangular prism or box with horizontal strata of cells for individuals, vertical strata parallel to the end planes for tests, and vertical strata parallel to the front plane for occasions. Such an arrangement will be termed a three-mode matrix. In general, an n-mode matrix would involve cross-classification of the data on n sets of indices, or modes. Each datum would correspond to an element of the Cartesian product of the sets of classification indices or modes.

Each mode will be identified by a lower-case letter, for example, the letter i may be used for the mode for individuals in a sample. It has proven convenient to use this lower case letter in several related, but distinct roles: 1) as a general identification of the mode, 2) as a subscript identifying the mode to which an element belongs, and 3) as a variable identification symbol for the elements in the mode. An example of the first usage is a statement such as "mode i is for the individuals in the sample." An example of the second usage is in the assignment of identification symbols $1_i, 2_i, 3_i, \cdots, N_{i_i}$ to the individuals in the sample. The identification symbol for each element of a mode is composed to two parts, one part being a number termed the index value of the element, designated by $v(i)$, and the other part being the identification subscript for mode. It will be noted that the elements for each mode constitute an ordered set. The index value will be utilized in any calculations for identification of elements. The series of index values for the elements in a mode shall consist of the integers from 1 to the number of elements in the mode. The number of elements in a mode will be designated by the captial letter N with the subscript identifying the mode, that is, by N_m where m is used in the present context as a generalized mode identification; thus, N_i is the number of elements in mode i. In the third role, the letter is used as a general, unspecified identification symbol which may be particularized to the identification symbol of each of the elements in turn. For example, x_{ijk} will be used as the generalized entry in the three-mode matrix

X with the letters i, j, and k being used as generalized identification symbols for the elements in the three modes. The index values for elements in a mode are designated by $v(m) = 1, 2, 3, \cdots, N_m$. The elements in a mode are designated by $m = 1_m, 2_m, 3_m, \cdots, N_{m_m}$.

A convenient notational step has been to define *combination modes* as contrasted to the *elementary modes* indicated in the preceding paragraph. A second-order combination mode is defined as the Cartesian product of two elementary modes and is denoted by the letters of the two elementary modes enclosed in parentheses. Thus, (ij) is a combination mode formed by the Cartesian product of the elementary modes i and j. Each element in the combination mode corresponds to a pair of elements from the elementary modes, one element of the pair from each of the two modes. Every such pair of elements from the two elementary modes corresponds to a distinct element in the combination mode. The index-value part of the identification symbol for each element in a combination mode is computed over the index values of the identification symbols of the corresponding paired elements of the elementary modes. The form of the equation is given below.

$$v(ij) = [v(i) - 1]N_j + v(j). \tag{1}$$

An illustration of this computation is given below for $N_i = 2$ and $N_j = 3$.

Identification Symbols for

Elementary Modes i	Elementary Modes j	Combination Mode (ij)
1_i	1_j	$1_{(ij)}$
1_i	2_j	$2_{(ij)}$
1_i	3_j	$3_{(ij)}$
2_i	1_j	$4_{(ij)}$
2_i	2_j	$5_{(ij)}$
2_i	3_j	$6_{(ij)}$

The order (ij) may be read as "i-outer loop, j-inner loop." This definition is compatible with computer calculation of subscripts. It is to be noted that (ij) and (ji) contain the same elements but with a change in order and corresponding index values. The number of elements in a combination mode is the product of the numbers of elements in the elementary modes:

$$N_{(ij)} = N_i N_j \,. \tag{2}$$

A matrix will be designated by a capital letter; an element in the matrix will be designated by a lower-case letter, with subscripts indicating the location of the element in the matrix. The first subscript will designate the row mode of a two-mode matrix or the horizontal strata mode of a three-mode matrix. The second subscript will designate the column mode for a two-mode matrix or the vertical strata mode parallel to the end planes of a three-mode matrix. The third subscript for an element of a three-mode matrix will designate the vertical strata mode parallel to the front plane of the matrix.

A notational device that has been found to be especially useful for two-mode matrices is to pre-subscript the letter for the matrix with the letter for the row mode and to post-subscript the matrix letter with the letter for the column mode. Thus, ${}_iA_m$ is the matrix having entries a_{im} , such as entry $a_{7_i 3_m}$ in row 7_i and column 3_m , with rows for mode i and columns for mode m. It is to be noted that the matrix ${}_mA_i$ with entries a_{mi} is the transpose of matrix ${}_iA_m$. The entry in matrix ${}_mA_i$ for elements 3_m of mode m and element 7_i of mode i is $a_{3_m 7_i}$; note that this entry is identical with entry $a_{7_i 3_m}$ in matrix ${}_iA_m$, the modes for rows and columns of the matrices having been interchanged as in transposing the matrix. Thus, for fixed values of i and m, a_{im} and a_{mi} are two ways of denoting the same quantity, the first in matrix ${}_iA_m$ and the second in the transposed matrix, ${}_mA_i$. This notational device will be utilized to designate the transpose of a matrix.

The use of subscripts here is at variance with common mathematical practice. Commonly, a matrix such as A is defined to have rows for one mode and columns for a second mode. The subscripts for the elements are used in common mathematical notation solely as indices; thus, i could be used as a row index to designate individual if the row mode were individuals and be used as a column index to indicate factor if the column mode were factors. In the first case i would be the first subscript and in the second case i would be the second subscript. In contrast, in the present notation, any given letter will designate a particular mode and the arrangement of the matrix will change with a change in location of the subscript. This change in notation permits the use of matrix letters to designate classes of matrices with the particular matrix in the class being designated by the modes involved. For example, A may be used to designate factor coefficients for individuals by factors. If m designates one set of factors and m^* designates another set of factors, the matrix ${}_iA_m$ will designate the coefficients for individuals i and factors m while the matrix ${}_iA_{m^*}$ will designate the coefficients for individuals i and factors m^*. As noted previously, the transpose of a matrix may be designated by interchange of the pre-subscript and the post-subscript. The letter A would still designate the class of matrix by the class of entries it contains.

Another convenience offered by the notation is in the representation of three-mode matrices. Let X be a three-mode matrix for modes i, j, and k with elements x_{ijk} . Consider use of the combination mode (ij). Then, the three-mode matrix may be written as a two-mode matrix with rows for the combination mode and columns for the third mode. This matrix could be denoted as ${}_{(ij)}X_k$. The same matrix can be written, also, as the two-mode matrix ${}_{(ik)}X_j$ by use of the combination mode (ik) for rows and mode j for columns. When the order of the elementary modes is considered along with the possible pairs of three things taken two at a time, that is, when the permutations of three things taken two at a time are considered, and when the possibility is considered that the combination variable can be used as the row mode or the column mode, there exist twelve ways that the three-mode matrix may be written as a two-mode matrix. The chosen notation is helpful in dealing with these cases.

In a matrix product of two-mode matrices, the post-subscript of the first matrix must conform with the pre-subscript of the second matrix. This matrix multiplication follows the usual convention of an entry of the product matrix equalling the sum of products between entries in a row of the first matrix with entries in a column of the second matrix. Consequently, the common subscript for the two matrices will be written only once. For example, the matrix product of matrices ${}_iA_j$ and ${}_jB_k$ will be written as ${}_iA_jB_k$. This product yields the matrix ${}_iC_k$ with row mode i and column mode k.

A matrix operation that has not been used extensively in psychometrics but which has been found especially helpful in the present context is the *direct product* or *Kronecker product* of two matrices. For a discussion of the direct product or Kronecker product see MacDuffee ([6], pages 81 ff.) or Bellman ([1], pages 226 ff.). Consider the two matrices ${}_iA_m$ and ${}_jB_p$ with elements a_{im} and b_{jp} . Let the entries of a matrix ${}_{(ij)}H_{(mp)}$ be $h_{(ij)(mp)}$ which are defined in terms of the entries of ${}_iA_m$ and ${}_jB_p$ by

$$h_{(ij)(mp)} = a_{im}b_{jp} . \tag{3}$$

The matrix notation for this operation will be

$$_{(ij)}H_{(mp)} = {}_iA_m \times {}_jB_p . \tag{4}$$

The Kronecker product matrix may be written as a supermatrix containing submatrices proportional to the matrix ${}_jB_p$, the second matrix of the product. The constants of proportionality are the elements of the first matrix ${}_iA_m$. Thus, rectangular representation of the Kronecker product matrix is given below.

$$_{(ij)}H_{(mp)} = \begin{bmatrix} (a_{1_i1_m}\ {}_jB_p) & (a_{1_i2_m}\ {}_jB_p) & (a_{1_i3_m}\ {}_jB_p) & \cdots \\ (a_{2_i1_m}\ {}_jB_p) & (a_{2_i2_m}\ {}_jB_p) & (a_{2_i3_m}\ {}_jB_p) & \cdots \\ (a_{3_i1_m}\ {}_jB_p) & (a_{3_i2_m}\ {}_jB_p) & (a_{3_i3_m}\ {}_jB_p) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

It is to be noted that the modes of the Kronecker product matrix are combination modes obtained from the modes of the matrices involved in the product. The row mode of the Kronecker product is obtained from the row modes of the matrices involved, the outer loop mode of the combination being the row mode for the first matrix and the inner loop being the row mode for the second matrix. The column mode of the Kronecker product is similarly a combination mode of the column modes of the matrices involved in the product.

Several propositions concerning the Kronecker product are important to the use of this operation in three-mode factor analysis. These propositions will be described here without proof. Bellman [1] discusses the less obvious propositions.

a) The transpose of a Kronecker product matrix equals the Kronecker product of the transposes of the original matrices in the same order as in the original product. Compare (6) with (4).

$$(6) \qquad {}_{(mp)}H_{(ij)} = {}_{m}A_{i} \times {}_{p}B_{j} \ .$$

b) If the matrices ${}_{i}A_{m}$ and ${}_{j}B_{p}$ are square and symmetric, then their Kronecker product ${}_{(ij)}H_{(mp)}$ will be a square, symmetric matrix.

c) If the matrices ${}_{i}A_{m}$ and ${}_{j}B_{p}$ are diagonal matrices, their Kronecker product will be a diagonal matrix containing products of pairs of diagonal elements, one member of the pair being a diagonal entry in ${}_{i}A_{m}$ and the other member of the pair being a diagonal entry in ${}_{j}B_{p}$, for all possible such pairs.

d) The proposition of (7) is true for Kronecker products.

$$(7) \qquad ({}_{i}A_{m}S_{n}) \times ({}_{j}B_{p}T_{q}) = ({}_{i}A_{m} \times {}_{j}B_{p})({}_{m}S_{n} \times {}_{p}T_{q}).$$

e) If matrices ${}_{i}A_{m}$ and ${}_{j}B_{p}$ possess left inverses, the left inverse of their Kronecker product is the Kronecker product of their left inverses in the same order; that is, if

$${}_{m}A_{i}^{\dagger}A_{m} = {}_{m}I_{m} \quad \text{and} \quad {}_{p}B_{j}^{\dagger}B_{p} = {}_{p}I_{p} \ ,$$

where ${}_{m}A_{i}^{\dagger}$ and ${}_{p}B_{j}^{\dagger}$ are the left inverses of ${}_{i}A_{m}$ and ${}_{j}B_{p}$ and where ${}_{m}I_{m}$ and ${}_{p}I_{p}$ are identity matrices, then

$$(8) \qquad ({}_{m}A_{i}^{\dagger} \times {}_{p}B_{j}^{\dagger})({}_{i}A_{m} \times {}_{j}B_{p}) = {}_{(mp)}I_{(mp)} \ ,$$

where ${}_{(mp)}I_{(mp)}$ is an identity matrix. Note that the left inverse matrix is denoted by a dagger and that the subscripts indicate row and column modes of this matrix.

f) If the matrices ${}_{i}A_{m}$ and ${}_{j}B_{p}$ are column-wise sections of orthonormal matrices, their Kronecker product will be a column-wise section of an orthonormal matrix. This proposition is a special case of the preceding proposition and occurs when ${}_{m}A_{i}^{\dagger}$ is the transpose of ${}_{i}A_{m}$ and ${}_{p}B_{j}^{\dagger}$ is the transpose of ${}_{j}B_{p}$.

g) The relation of the characteristic roots and vectors of two square, symmetric matrices to the roots and vectors of their Kronecker product is of considerable importance. Let the two matrices be ${}_{j}P_{j}$ and ${}_{k}Q_{k}$ and let their Kronecker product be ${}_{(jk)}S_{(jk)}$ as given in (9).

(9) $$ {}_iP_i \times {}_kQ_k = {}_{(ik)}S_{(ik)} \ . $$

Let the matrices ${}_iP_i$ and ${}_kQ_k$ be resolved into their characteristic roots and vectors as in (10).

(10) $$ {}_iP_i = {}_iB_pP_pB_i \quad \text{and} \quad {}_kQ_k = {}_kC_qQ_qC_k \ , $$

where the matrices ${}_pP_p$ and ${}_qQ_q$ are diagonal matrices containing the characteristic roots of ${}_iP_i$ and ${}_kQ_k$, respectively, and the matrices ${}_iB_p$ and ${}_kC_q$ contain, as column vectors, the characteristic vectors of ${}_iP_i$ and ${}_kQ_k$, respectively. The characteristic vectors are unit vectors so that

(11) $$ {}_pB_iB_p = {}_pI_p \quad \text{and} \quad {}_qC_kC_q = {}_qI_q \ . $$

Let the matrix ${}_{(ik)}S_{(ik)}$ be resolved into its characteristic roots and vectors as in (12).

(12) $$ {}_{(ik)}S_{(ik)} = {}_{(ik)}V_{(pq)}S_{(pq)}V_{(ik)} \ , $$

where ${}_{(pq)}S_{(pq)}$ is a diagonal matrix containing the characteristic roots and ${}_{(ik)}V_{(pq)}$ contains, as column vectors, the characteristic vectors of ${}_{(ik)}S_{(ik)}$. The characteristic vectors are unit vectors so that

(13) $$ {}_{(pq)}V_{(ik)}V_{(pq)} = {}_{(pq)}I_{(pq)} \ . $$

The interesting relations are given by (14) and (15).

(14) $$ {}_{(pq)}S_{(pq)} = {}_pP_p \times {}_qQ_q \ . $$

(15) $$ {}_{(ik)}V_{(pq)} = {}_iB_p \times {}_kC_q \ . $$

These relations are developed from substitution from (10) into (9) and use of propositions (c) and (f). Thus, the matrix of characteristic roots of the Kronecker product matrix is the Kronecker product of the matrices of characteristic roots of the two matrices involved in the product; and the matrix of characteristic vectors of the Kronecker product matrix is the Kronecker product of the matrices of characteristic vectors of the matrices that enter into the product.

Fundamental Model for Three-Mode Factor Analysis

A body of data to which three-mode factor analysis would be applied can be recorded in a three-mode matrix X which has cell entries x_{ijk} . One example of such a body of data was used as an illustration in the section *Remarks on Notation* and consisted of scores of a sample of individuals on a battery of tests on several occasions. A second example could be the ratings of a sample of individuals on a selection of traits by several raters. A third example could be ratings by a sample of individuals on a selection of bipolar adjective scales of a selection of concepts as is done in studies using the semantic differential developed by Osgood, Suci, and Tannenbaum [7]. In each of these cases, the data consist of numerical values which are identified by three modes of classification. These modes are directly related to the observation of the data and may be termed observational modes. These observational modes will be designated as mode i, mode j, and mode k.

A convenient form in which to list the data is as a two-mode matrix with elementary mode i for rows and combination mode (jk) for columns. This is matrix ${}_iX_{(jk)}$. Table 1 gives observed scores in a matrix ${}_iX_{(jk)}$ for a fictitious example that will be used to illustrate these notes.

Since, in practice, an allowance should be made for discrepancies in fitting a model to observed data, such an allowance is symbolized in the present context by letting $\hat{x}_{ijk}$ be the value obtained from the model and e_{ijk} be the value of the discrepancy. Then

$$x_{ijk} = \hat{x}_{ijk} + e_{ijk} . \tag{16}$$

It is desirable, of course, that the discrepancies should be very small. In the present conception, a major contribution to $\hat{x}_{ijk}$ will be made by a component labeled $\bar{x}_{ijk}$ to be discussed in this section. Other components of limited scope and having very limiting definition may be added to the $\bar{x}_{ijk}$ to produce the $\hat{x}_{ijk}$. Such a component will be introduced in the last section of this paper. For theoretical purposes in the present section, the conceptual measures $\bar{x}_{ijk}$ will treated as if known. Later sections will treat problems of fitting the model to data.

The model for $\bar{x}_{ijk}$, written in summational notation, is

$$\bar{x}_{ijk} = \sum_m \sum_p \sum_q a_{im} b_{jp} c_{kq} g_{mpq} . \tag{17}$$

In this model, three derivational modes, m, p, and q, are defined as conceptually more basic than the modes employed in making the observations. Each of these derivational modes corresponds to one of the observational modes: m corresponding to i, p corresponding to j, and q corresponding to k. Each of these derivational modes can be thought of as a set of factors in the domain of the corresponding observational mode. An alternate interpretation is to think of each derivational mode as consisting of conceptual, or idealized categories corresponding to the observational mode. Thus, if the observational mode i is used to designate individuals in a sample, the derivational mode m can be thought of consisting of factors among individuals or of conceptual, or idealized individuals. It is hoped that the number of elements in each derivational mode will be markedly less than the number of elements in the corresponding observational mode. This hope is prefaced by the condition that a sufficiently large number of elements are included in each observational mode from the domain of elements that might be included.

The coefficients a_{im} , b_{jp} , and c_{kq} are entries in two-mode matrices ${}_iA_m$, ${}_jB_p$, and ${}_kC_q$. These coefficients describe the elements in the observational modes in terms of the elements in the derivational modes. The coefficients g_{mpq} are entries in the three-mode matrix G which is termed the "core matrix." Just as in the original three-mode matrix, X, in which each cell represents a particular combination of categories from the observational modes and the entry is a measure of a phenomenon whose value depends on the combination of categories, in the same way each cell in the core matrix, G, represents a unique combination of categories from the derivational modes and the entry is a measure of the phenomenon for this combination of categories. The core matrix can be thought of as describing the basic relations existent in the measures of the phenomenon being observed. The two-mode matrices ${}_iA_m$, ${}_jB_p$, and ${}_kC_q$ transform the statements of these relations from applying to the more basic derivational modes to applying to the observational modes. The interrelations among elements of one of the observational modes depend, in part, on the similarity of their relations to the derivational modes, and, in part, to the relations in the core matrix.

The fundamental model given in (17) can be written in terms of two-mode matrices by use of combination variables and Kronecker products. Three such interpretations are

$$ {}_i\tilde{X}_{(jk)} = {}_iA_m G_{(pq)}({}_pB_j \times {}_qC_k), \tag{18a} $$

$$ {}_j\tilde{X}_{(ik)} = {}_jB_p G_{(mq)}({}_mA_i \times {}_qC_k), \tag{18b} $$

$$ {}_k\tilde{X}_{(ij)} = {}_kC_q G_{(mp)}({}_mA_i \times {}_pB_j). \tag{18c} $$

These three forms involve using each of the observational modes as the row mode for the matrix $\tilde{X}$ and the other two observational modes as a combination mode of the two-mode matrix. The core matrix G is correspondingly arranged to the matrix $\tilde{X}$ in each of these equations, taking into account the correspondence of observational modes to derivational modes. In each of these equations, the elementary derivational mode used as the row mode for the matrix G is transformed by the appropriate two-mode matrix to the observational mode used as the row mode of matrix $\tilde{X}$. For example, in (18a), the two-mode matrix ${}_iA_m$ transforms the row mode m of matrix G to the row mode i of matrix $\tilde{X}$.

An interesting point in (18) is the natural way the use of combination modes in writing three-mode matrices as two-mode matrices is conformal with the use of the Kronecker product. The development of these equations will be illustrated with the development of (18a) from (17) which can be rewritten as

$$ \tilde{x}_{ijk} = \sum_m a_{im} \sum_p \sum_q g_{mpq}(b_{jp}c_{kq}). \tag{19} $$

When the portion of the right term of (19) involving the double summation over p and q is written as a two-mode matrix with rows for mode m and columns for the combination mode (jk), the sum of products over mode m with a_{im} can be written as the first matrix multiplication on the right side of (18a). It is to be noted that the product $(b_{ip}c_{kq})$ is an element of the Kronecker product $({}_pB_i \times {}_qC_k)$ as per (3) and (4) with appropriate arrangement of the subscripts. The matrix multiplication ${}_mG_{(pq)}({}_pB_i \times {}_qC_k)$ accomplishes the double summation over p and q. Equations (18b) and (18c) are similarly developed.

Some matrix rank restrictions are convenient for the present discussions. These restrictions may be relaxed in some restricted senses in future developments; however, they will be assumed to be satisfied for the present.

(20a) $\text{Rank}\ ({}_i\tilde{X}_{(jk)}) = \text{Rank}\ ({}_iA_m) = \text{Rank}\ ({}_mG_{(pq)}({}_pB_j \times {}_qC_k)) = N_m\ .$

(20b) $\text{Rank}\ ({}_j\tilde{X}_{(ik)}) = \text{Rank}\ ({}_jB_p) = \text{Rank}\ ({}_pG_{(mq)}({}_mA_i \times {}_qC_k)) = N_p\ .$

(20c) $\text{Rank}\ ({}_k\tilde{X}_{(ij)}) = \text{Rank}\ ({}_kC_q) = \text{Rank}\ ({}_qG_{(mp)}({}_mA_i \times {}_pB_j)) = N_q\ .$

For justification of these restrictions, consider the first equation, (20a). A possibility exists that the rank of ${}_iA_m$ is less than its column order. In this case ${}_iA_m$ could be post-multiplied by a nonsingular transformation matrix coupled with pre-multiplication of ${}_mG_{(pq)}$ by the inverse of the transformation matrix. Such a transformation will not alter the product ${}_iA_mG_{(pq)}$. This transformation may be selected so that one or more columns of the transformed ${}_iA_m$ contain all zero entries. The number of columns containing nonzero entries in the transformed ${}_iA_m$ could equal the rank of ${}_iA_m$, but not be less than the rank of ${}_iA_m$. Those columns of the transformed ${}_iA_m$ containing all zero entries can be discarded along with the corresponding rows of the transformed ${}_mG_{(pq)}$, thus reducing the column order of ${}_iA_m$ and row order of ${}_mG_{(pq)}$ to the rank of ${}_iA_m$. It is assumed in the specification in (20a) that any such order reduction possibilities have been accomplished and the possibility of the rank of ${}_iA_m$ being less than N_m is discarded. A corresponding argument applies to the case for which the rank of $[{}_mG_{(pq)}({}_pB_j \times {}_qC_k)]$ is less than the number of rows, N_m . Note that, if the ranks of ${}_iA_m$ and $[{}_mG_{(pq)}({}_pB_j \times {}_qC_k)]$ are N_m , then the rank of ${}_i\tilde{X}_{(jk)}$ is N_m also. Thus, all possibilities violating the restrictions are discarded.

The three ranks given for the matrix $\tilde{X}$ when written in three different ways are not necessarily equal. They are connected by the interesting group of inequalities that state that no one of the three ranks can be greater than the product of the other two ranks. This statement results from considering the matrix G written in each of the three ways included in (18) and (20). Note that the number of rows of G when written in each way equals the rank of $\tilde{X}$ written in the corresponding way. The number of columns of G is the product of the ranks of the matrix $\tilde{X}$ written in the two non-corresponding ways. For example, the number of rows of ${}_mG_{(pq)}$ is N_m which is the rank of ${}_i\tilde{X}_{(jk)}$ and the number of columns is the product of N_p and N_q which are the ranks of the matrices ${}_j\tilde{X}_{(ik)}$ and ${}_k\tilde{X}_{(ij)}$. If the number of rows of G written in any of the three ways were greater than the number of columns, then the rank of that way of writing G would be equal to or less than the number of columns and would surely be less than the number of rows. However, the rank of G written in any of the three ways must equal the number of rows of that way of writing G. Therefore, no one of the three values N_m , N_p , and N_q can be greater than the product of the other two.

An interesting group of relations may be developed from the products defined by

(21a) $${}_i\tilde{M}_i = {}_i\tilde{X}_{(jk)}\tilde{X}_i \ ,$$

(21b) $${}_j\tilde{P}_j = {}_j\tilde{X}_{(ik)}\tilde{X}_j \ ,$$

(21c) $${}_k\tilde{Q}_k = {}_k\tilde{X}_{(ij)}\tilde{X}_k \ .$$

Substitution from (18) into (21) yields

(22a) $${}_i\tilde{M}_i = {}_iA_mG_{(pq)}({}_pB_j \times {}_qC_k)({}_jB_p \times {}_kC_q)_{(pq)}G_mA_i \ ,$$

(22b) $${}_j\tilde{P}_j = {}_jB_pG_{(mq)}({}_mA_i \times {}_qC_k)({}_iA_m \times {}_kC_q)_{(mq)}G_pB_j \ ,$$

(22c) $${}_k\tilde{Q}_k = {}_kC_qG_{(mp)}({}_mA_i \times {}_pB_j)({}_iA_m \times {}_jB_p)_{(mp)}G_qC_k \ .$$

Note the use of the transpose of a Kronecker product as indicated in (6). Let

(23a) $${}_mM_m = {}_mG_{(pq)}({}_pB_j \times {}_qC_k)({}_jB_p \times {}_kC_q)_{(pq)}G_m \ ,$$

(23b) $${}_pP_p = {}_pG_{(mq)}({}_mA_i \times {}_qC_k)({}_iA_m \times {}_kC_q)_{(mq)}G_p \ ,$$

(23c) $${}_qQ_q = {}_qG_{(mp)}({}_mA_i \times {}_pB_j)({}_iA_m \times {}_jB_p)_{(mp)}G_q \ .$$

Then from (22) and (23)

(24a) $${}_i\tilde{M}_i = {}_iA_mM_mA_i \ ,$$

(24b) $${}_j\tilde{P}_j = {}_jB_pP_pB_j \ ,$$

(24c) $${}_k\tilde{Q}_k = {}_kC_qQ_qC_k \ .$$

The form of (24) suggests that the matrices ${}_iA_m$, ${}_jB_p$, and ${}_kC_q$ could be determined as factor matrices of the product matrices with the matrices ${}_mM_m$, ${}_pP_p$, and ${}_qQ_q$ being analogous to covariance matrices among the factors. This is, in fact, a legitimate interpretation. Any method of factoring that produces factor matrices in which the number of columns is equal to the rank of the matrix being factored may be used. Application of principal axes factoring will be discussed in a subsequent section. Before discussing this topic further, it is necessary to consider possible transformations of the derivational modes.

Let the matrices ${}_mT_{m*}$, ${}_pT_{p*}$, and ${}_qT_{q*}$ be square, nonsingular matrices and let

(25a) $$ {}_iA_m T_{m^*} = {}_iA_{m^*} , $$

(25b) $$ {}_jB_p T_{p^*} = {}_jB_{p^*} , $$

(25c) $$ {}_kC_q T_{q^*} = {}_kC_{q^*} . $$

The m^*, p^*, and q^* are transformed derivational modes and the matrices ${}_iA_{m^*}$, ${}_jB_{p^*}$, and ${}_kC_{q^*}$ contain coefficients describing the observational mode elements in terms of the transformed derivational modes. The inverse transformations are

(26a) $$ {}_iA_{m^*}({}_mT_{m^*})^{-1} = {}_iA_m , $$

(26b) $$ {}_jB_{p^*}({}_pT_{p^*})^{-1} = {}_jB_p , $$

(26c) $$ {}_kC_{q^*}({}_qT_{q^*})^{-1} = {}_kC_q . $$

It is to be noted the row modes and column modes must be interchanged when inverting a matrix. Substitution from (26) into (18a) yields

(27a) $$ {}_i\tilde{X}_{(jk)} = {}_iA_{m^*}({}_mT_{m^*})^{-1}{}_mG_{(pq)}[({}_{p^*}T_p)^{-1}{}_{p^*}B_j \times ({}_{q^*}T_q)^{-1}{}_{q^*}C_k]. $$

Use of the proposition of (7) concerning Kronecker products yields

(28a) $$ {}_iX_{(jk)} = {}_iA_{m^*}({}_mT_{m^*})^{-1}{}_mG_{(pq)}[({}_{p^*}T_p)^{-1} \times ({}_{q^*}T_q)^{-1}]({}_{p^*}B_j \times {}_{q^*}C_k). $$

Let

(29a) $$ {}_{m^*}G_{(p^*q^*)} = ({}_mT_{m^*})^{-1}{}_mG_{(pq)}[({}_{p^*}T_p)^{-1} \times ({}_{q^*}T_q)^{-1}], $$

(29b) $$ {}_{p^*}G_{(m^*q^*)} = ({}_pT_{p^*})^{-1}{}_pG_{(mq)}[({}_{m^*}T_m)^{-1} \times ({}_{q^*}T_q)^{-1}], $$

(29c) $$ {}_{q^*}G_{(m^*p^*)} = ({}_qT_{q^*})^{-1}{}_qG_{(mp)}[({}_{m^*}T_m)^{-1} \times ({}_{p^*}T_p)^{-1}]. $$

Substitution of (29a) into (28a) yields

(30a) $$ {}_i\tilde{X}_{(jk)} = {}_iA_{m^*}G_{(p^*q^*)}({}_{p^*}B_j \times {}_{q^*}C_k). $$

Equation (29a) gives the transformed G matrix that is developed in steps involving (27a), (28a), and similar statements may be made for the other two modes. Equation (30a) indicates that the use of the transformed two-mode coefficient matrices and transformed core matrix G reproduces the model in the form of (18a). The other two equations of group (30) would demonstrate the same point.

Equations (29) yield the same transformed G matrix written in the three ways as two-dimensional matrices. Once the transformation matrices are determined by some rotation of axes procedure, the transformed G matrix can be determined by some one of the set (29).

Tracing through the effects of the transformations on the matrices ${}_mM_m$, ${}_pP_p$, and ${}_qQ_q$ produces the following transformed matrices to maintain the form of (23) and (24) for the transformed coefficient matrices and core matrix.

(31a) $$_{m*}M_{m*} = ({}_mT_{m*})^{-1}{}_mM_m({}_{m*}T_m)^{-1}.$$

(31b) $$_{p*}P_{p*} = ({}_pT_{p*})^{-1}{}_pP_p({}_{p*}T_p)^{-1}.$$

(31c) $$_{q*}Q_{q*} = ({}_qT_{q*})^{-1}{}_qQ_q({}_{q*}T_q)^{-1}.$$

The existence of the flexibility afforded by the freedom of transformation permitted by the model is both important and the source of many problems. There is a lack of uniqueness. This gives rise to many problems yet to be solved. It is important to recognize and to remember this lack of uniqueness and the resultant freedom of transformation.

One example of the utilization of the freedom of transformation is in choice of factoring methods to be applied to the product matrices ${}_i\tilde{M}_i$, ${}_j\tilde{P}_j$, and ${}_k\tilde{Q}_k$. For example, let the transformations be so chosen that ${}_{m*}M_{m*}$ is an identity matrix. Then the transformed (24a) would be

$${}_i\tilde{M}_i = {}_iA_{m*}A_i \ .$$

This is in standard form for many of the methods of factor analysis, such as principal axes, centroid, and square-root. Each method implies a different transformation.

Determination of the matrices ${}_pT_{p*}$ and ${}_qT_{q*}$ by rotation of matrices ${}_jB_p$ and ${}_kC_q$ to simple structure has seemed to be successful for several studies. In these studies, the mode i has been used to represent individuals. Either the matrix ${}_iA_m$ was not determinate as discussed in a subsequent section or the rotation to simple structure was not markedly successful. An alternate method of rotation for determining the ${}_mT_{m*}$ matrix was to rotate the core matrix to simple structure. This met with moderate success. While the formal aspects of the transformation of derivational modes are indicated in the preceding discussion, the practical problems of determination of these transformations are yet to be solved.

Up to this point, the discussion has involved the relation of three-mode matrix $\tilde{X}$ to the core matrix G. The inverse relation will be discussed in this paragraph. In this discussion, the left inverses of the matrices ${}_iA_m$, ${}_jB_p$, and ${}_kC_q$ will be used. These left inverses are defined by

(32a) $${}_mA_i^{\dagger} = ({}_mA_iA_m)^{-1}{}_mA_i \ ,$$

(32b) $${}_pB_j^{\dagger} = ({}_pB_jB_p)^{-1}{}_pB_j \ ,$$

(32c) $${}_qC_k^{\dagger} = ({}_qC_kC_q)^{-1}{}_qC_k \ .$$

The existence of these inverses is assured by the conditions placed on the ranks of the matrices in (20). Proposition (e) as to Kronecker products yields left inverses of the Kronecker products of the coefficient matrices. Use of these inverses in (18) yields

(33a) $${}_mA_i^{\dagger}\tilde{X}_{(jk)}({}_jB_p^{\dagger} \times {}_kC_q^{\dagger}) = {}_mG_{(pq)} \ ,$$

(33b) $$ {}_pB_j^\dagger \tilde{X}_{(ik)}({}_iA_m^\dagger \times {}_kC_q^\dagger) = {}_pG_{(mq)} , $$

(33c) $$ {}_qC_k^\dagger \tilde{X}_{(ij)}({}_iA_m^\dagger \times {}_jB_p^\dagger) = {}_qG_{(mp)} . $$

Equations (33) reinforce the idea that the matrices $\tilde{X}$ and G are related by linear transformation by two-mode matrices of coefficients for each of the three corresponding pairs of modes, i with m, j with p, and k with q. It appears reasonable to say that the core matrix G is a transformation of the matrix $\tilde{X}$. The reverse statement is proper also.

The following inverse relations may be noted for the product matrices ${}_i\tilde{M}_i$, ${}_j\tilde{P}_j$, and ${}_k\tilde{Q}_k$.

(34a) $$ {}_mA_i^\dagger \tilde{M}_i A_m^\dagger = {}_mM_m , $$

(34b) $$ {}_pB_j^\dagger \tilde{P}_j B_p^\dagger = {}_pP_p , $$

(34c) $$ {}_qC_k^\dagger \tilde{Q}_k C_q^\dagger = {}_qQ_q . $$

These equations are obtained from (24).

A group of important structural relations occur when the coefficient matrices are column-wise sections of orthonormal matrices; that is, when

(35) $$ {}_mA_i A_m = {}_mI_m \ ; \quad {}_pB_j B_p = {}_pI_p \ ; \quad {}_qC_k C_q = {}_qI_q . $$

Then the left inverses defined in (32) are the transposes of the coefficient matrices. Then, the Kronecker product of pairs of these coefficient matrices are also column-wise sections of orthonormal matrices, as per proposition (f) concerning Kronecker products.

Refer to (23). The products of Kronecker products on the right-hand side of these equations are identity matrices for this case and can be deleted so that these equations become

(36a) $$ {}_mM_m = {}_mG_{(pq)}G_m , $$

(36b) $$ {}_pP_p = {}_pG_{(mq)}G_q , $$

(36c) $$ {}_qQ_q = {}_qG_{(mp)}G_q . $$

It is interesting to note that the matrices ${}_mM_m$, ${}_pP_p$, and ${}_qQ_q$ are product matrices of the core matrix G written in the three ways. Thus, (24) and (34) give transformations between corresponding product matrices for $\tilde{X}$ and G.

Due to the rank statements of (20) and to the coefficient matrices being column-wise sections of orthonormal matrices, the traces of corresponding pairs of these product matrices are equal.

(37a) $$ \text{tr}\,({}_i\tilde{M}_i) = \text{tr}\,({}_mM_m), $$

(37b) $$ \text{tr}\,({}_j\tilde{P}_j) = \text{tr}\,({}_pP_p), $$

(37c) $$ \text{tr}\,({}_k\tilde{Q}_k) = \text{tr}\,({}_qQ_q), $$

where, for example, tr $({}_i\tilde{M}_i)$ denotes the trace of matrix ${}_i\tilde{M}_i$, which is the sum of the diagonal entries of this matrix. Note that the diagonal entries in each product matrix are the sum of squares of the entries in the corresponding rows of the matrix forming the product in (21) or (36). Therefore, the trace of each product matrix equals the sum of squares of the entries in the entire matrix forming that product matrix. A consequence of these relations is that the sum of squares of the entries in the matrix $\tilde{X}$ equals the sum of squares of the entries in the core matrix G.

$$\sum_i \sum_j \sum_k x_{ijk}^2 = \sum_m \sum_p \sum_q g_{mpq}^2 . \tag{38}$$

This is a most interesting relation between these two matrices.

Some further important relations for the case when the coefficient matrices are column-wise sections of orthonormal matrices occur when the transformations from matrix $\tilde{X}$ to matrix G are taken in steps. Let

$$ {}_m\tilde{X}_{(jk)} = {}_mA_i\tilde{X}_{(jk)} = {}_mG_{(pq)}({}_pB_j \times {}_qC_k), \tag{39}$$

where the second equation is obtained from (18a). A rewritten form of (39) follows where there is a change in the formation of the combination variables used as column modes for the two-mode matrices.

$$ {}_j\tilde{X}_{(mk)} = {}_j\tilde{X}_{(ik)}({}_iA_m \times {}_kI_k) = {}_jB_pG_{(mq)}({}_mI_m \times {}_qC_k). \tag{40}$$

Note that

$$ {}_j\tilde{X}_{(mk)}\tilde{X}_j = {}_j\tilde{X}_{(ik)}({}_iA_m \times {}_kI_k)({}_mA_i \times {}_kI_k)_{(ik)}\tilde{X}_j .$$

When it is noted that the Kronecker product on the right-hand side is a section of an orthonormal matrix so that the product of Kronecker products produces an identity matrix which may be deleted, then

$$ {}_j\tilde{X}_{(mk)}\tilde{X}_j = {}_j\tilde{X}_{(ik)}\tilde{X}_j = {}_j\tilde{P}_j . \tag{41}$$

Thus, the product matrix ${}_j\tilde{P}_j$ defined in (21b) is not affected by an orthonormal transformation applied to the mode i. The same conclusion could be obtained by an orthonormal transformation on mode k. A more general form of this relation is obtained by defining matrices of the form

$$ {}_k\tilde{X}_{(mp)} = {}_k\tilde{X}_{(mj)}({}_mI_m \times {}_jB_p) = {}_k\tilde{X}_{(ij)}({}_iA_m \times {}_jB_p). \tag{42}$$

It can be shown by steps like those between (40) and (41) that

$$ {}_k\tilde{X}_{(mp)}\tilde{X}_k = {}_k\tilde{X}_{(ij)}\tilde{X}_k = {}_k\tilde{Q}_k . \tag{43}$$

The general proposition is that the product matrix for any one mode is invariant over orthonormal transformations applied to other modes of a three-mode matrix. Use of this proposition will be indicated in the next section on factoring a three-mode matrix by an extension of the method of principal axes.

Factoring by Characteristic Roots and Vectors

The preceding section has treated the fundamental structure of the model employed in three-mode factor analysis. In this section and the next, the focus of attention will be on operational matters of procedures for fitting the model to observations. The particular procedures to be described will be based on determination of characteristic roots and vectors of various product matrices. These procedures are an outgrowth of the principal components and principal axes factoring procedures commonly applied in two-mode factor analysis. They are based also on the development by Eckart and Young [3] on the approximation of one matrix by another of lower rank. In the present section, the $\bar{x}_{ijk}$ will constitute the $\hat{x}_{ijk}$ of (16).

One matter considered as external to three-mode factor analysis, but of considerable importance to the analysis for each body of data is the scaling of those data. The model is written in terms of sums of squares and sums of products of the observations. If a reasonable origin of measurement exists for a particular body of data, the experimenter may find it preferable not to use deviation scores but to use the original measures. In this case, the model would be approximating the original observations and not just the deviations from some mean. In another case, it may be reasonable to consider that all measures for one of the variables in a mode such as j involve a single scale of measurement irrespective of the elements of the other modes. However, this scale may involve an arbitrary origin and unit of measure. Then, the measures for this variable might be transformed to standard scores over all measures for this variable. For example, suppose that the data consisted of scores of a sample of individuals, mode i, on tests in a battery, mode j, given on several occasions, mode k. A distribution of scores could be tabulated on each test in which the score of each individual on each occasion was entered separately from his score on each other occasion. Thus, each individual would be represented by N_k scores on this distribution. All scores on the distribution could be transformed to deviation scores with unit variance.

A third alternative exists when the measurement for each variable in mode j is considered to be on a different scale for each occasion or situation in mode k. Then the scores for combination variables (jk) could be separately standardized. In general, decisions on scaling of each particular body of data should be considered in terms of characteristics of the data. The model is not particularized to a single type of scaling of the data.

For many applications of three-mode factor analysis, one mode of the data will consist of individuals in a sample. In these cases mode i may be used to represent the sample with N_i being the number of individuals in the sample. It appears to be desirable to consider all scores divided by $\sqrt{N_i}$ before entering it into the three-mode factor analysis. Suppose that y_{ijk} represents the data appropriately scaled, for example, as standard scores for each variable. Then, let

(44)
$$x_{ijk} = \frac{1}{\sqrt{N_i}} y_{ijk} .$$

This is done so that matrices such as ${}_jP_j$ contain mean squares and mean products in the sense given below.

(45)
$$ {}_jP_j = {}_jX_{(ik)}X_j = \frac{1}{N_i} {}_jY_{(ik)}Y_j .$$

If product matrices are computed for the scores y_{ijk}, these product matrices should be divided by N_i to produce the product matrices for scores x_{ijk}.

The general strategy of the procedures to be discussed is to develop transformations on the observed data so that the following properties hold.

1) The transformations are of the form of the three-mode factor analysis model.

2) These transformations are column-wise sections of orthonormal matrices.

3) The contribution of successive elements of the derivational modes to the total sum of squares is in decreasing order.

4) The coefficient matrices along with the core matrix account in full for the observed data.

Then, the approximation is developed by truncation of each derivational mode in such a way as to retain only those elements that make nontrivial contributions to the sum of squares of the observed data. The foregoing statement is not as strong as corresponding statements for principal component factoring of two-mode matrices. Even though the computations in the following procedures involve steps analogous to steps in principal components analysis, these procedures do not produce a least-squares approximation to the data. Investigations of the mathematics of a least-squares fit for three-mode factor analysis indicate a need for an involved series of successive approximations. The strict least-squares fit to the data will not be considered in this report.

In order to specify the strategy in more detail, the following observed product matrices are defined.

(46a)
$$ {}_iM_i = {}_iX_{(jk)}X_i ,$$

(46b)
$$ {}_jP_j = {}_jX_{(ik)}X_j ,$$

(46c)
$$ {}_kQ_k = {}_kX_{(ij)}X_k .$$

Consider the characteristic roots and vectors of these observed product matrices and let the modes for these dimensions be m_2, p_2, and q_2 with N_{m_2}, N_{p_2}, and N_{q_2} as the number of nonzero roots. Note that all roots of product matrices will be diagonal entries in the diagonal matrices ${}_{m_2}M_{m_2}$,

${}_{p_2}P_{p_2}$, and ${}_{q_2}Q_{q_2}$. Also, let the roots for each matrix be arranged in descending order. Further, let the vectors corresponding to the roots be entered as columns in the matrices ${}_iA_{m_2}$, ${}_jB_{p_2}$, and ${}_kC_{q_2}$. Then

(47a) $${}_iM_i = {}_iA_{m_2}M_{m_2}A_i ,$$

(47b) $${}_jP_j = {}_jB_{p_2}P_{p_2}B_j ,$$

(47c) $${}_kQ_k = {}_kC_{q_2}Q_{q_2}C_k .$$

The core matrix may be obtained by a formula analogous to (33a), remembering that the coefficient matrices are column-wise sections of orthonormal matrices so that their left inverses are their transposes.

(48) $${}_{m_2}G_{(p_2q_2)} = {}_{m_2}A_i X_{(jk)}({}_jB_{p_2} \times {}_kC_{q_2}).$$

Since all nonzero roots are retained, a precise fit to the observed data matrix ${}_iX_{(jk)}$ is obtained by an extension of (18a).

(49) $${}_iX_{(jk)} = {}_iA_{m_2}G_{(p_2q_2)}({}_{p_2}B_j \times {}_{q_2}C_k).$$

The foregoing analysis could be termed the complete model for the observed data and all of the propositions developed in the preceding section could be applied.

According to (36a) each root in matrix ${}_{m_2}M_{m_2}$ is a sum of squares of entries in a horizontal plane in the core matrix G. Similarly by (36b) and (36c), each root in matrix ${}_{p_2}P_{p_2}$ is the sum of squares of entries in a vertical plane of G parallel to the end planes and each root in ${}_{q_2}Q_{q_2}$ is the sum of squares of entries in a vertical plane parallel to the front plane of G. Since the roots are arranged in descending order in each matrix, the lower planes, the right planes, and the rear planes should contain small entries. Consequently, some of these planes could be deleted without markedly reducing the total sum of squares of the entries in the matrix G. It was shown in the discussion associated with (37) that the sum of squares of the X matrix for any given G matrix equalled the sum of squares of the G matrix provided, as in the present case, that the coefficient matrices were column-wise sections of orthonormal matrices. It is proposed to define the approximation model by truncating the modes m_2, p_2, and q_2 by deleting elements corresponding to small roots. The modes for the approximation, then, will contain the first N_m, N_p, and N_q elements from the modes from the complete model. No precise way has been developed for the decision as to what elements should be retained in the approximation model. This is the same problem as the number of factors problem in two-mode factor analysis.

A major problem in the foregoing determination of the approximation model stems from the fact that each entry in the G matrix is in three planes. Thus, an entry may contribute to the sum of squares for one, two, or three small roots, one root being from each of the modes. This fact makes the sum of squares of the entries dropped by the truncation procedure applied to the three modes less than the sum of the roots dropped. A further problem generated by these interdependencies between the modes is that the deletion of elements for one mode results in the product matrices for the other modes for the reduced G matrix no longer to be diagonal, and thus not be associated with characteristic roots and vectors solutions for the $\tilde{X}$ matrix. It is for this reason that the proposed procedure does not yield, necessarily, a least-squares approximation to the observed data.

A point that will not be considered here is the transformation of the derivational modes. This topic, which is analogous to the rotation of axes problem in two-mode factor analysis, was discussed in association with (25) through (31). Such transformations should be considered in the analysis of each body of data.

Following are notes on three operational procedures utilizing the strategy discussed in the foregoing paragraphs.

Method I

This method, which follows directly from the preceding discussion, is of limited application since it involves all three observed product matrices directly. When one of the observational modes is relatively large, the corresponding product matrix is large also and may exceed computer capacity. For example, if the data were collected on a sample of 300 subjects, the product matrix ${}_iM_i$ would be 300 × 300. The solution for characteristic roots and vectors for this matrix would not be feasible. However, this method has been used on smaller bodies of data and is outlined here.

1) Compute the product matrices ${}_iM_i$, ${}_jP_j$, and ${}_kQ_k$ from the observed data matrix X as per (46).

2) Compute the characteristic roots and vectors of these product matrices.

3) Retain only the roots considered to be significant in some sense and form the diagonal matrices ${}_mM_m$, ${}_pP_p$, and ${}_qQ_q$ containing the retained roots in descending order. As indicated in the general discussion, the problem of how to arrive at the decisions as to which roots to retain has not been solved. The best procedure available may be to make the plot between root number and root size for each of the product matrices and to inspect the resulting series of points for a break from a steep slope to a more gentle slope and to retain all roots preceding this break. In any case, small roots should be discarded.

4) Form the coefficient matrices ${}_iA_m$, ${}_jB_p$, and ${}_kC_q$ containing as columns the unit-length characteristic vectors corresponding to the roots retained.

5) Compute the core matrix G by one of the equations like

$$(50) \qquad {}_mA_i X_{(jk)} ({}_jB_p \times {}_kC_q) = {}_mG_{(pq)} \ ,$$

which is obtained from (48). An alternate procedure is to make a series of computations such as listed below. Note that the output three-mode matrix from each step is rewritten for the next step as to row and column modes.

(51) $$_{m}A_{i}X_{(jk)} = {}_{m}X_{(jk)} \,.$$

(52) $$_{p}B_{j}X_{(mk)} = {}_{p}X_{(mk)} \,.$$

(53) $$_{q}C_{k}X_{(mp)} = {}_{q}G_{(mp)} \,.$$

Method II

The problem of one of the product matrices being too large for feasible computation by Method I may be solved by use of Method II. In describing this method it is assumed that the large mode is mode i for individuals. Following is an outline of the steps in this method.

1) Compute the product matrix $_{j}P_{j}$ for the observed matrix X as per (46b).

2) Determine the characteristic roots and vectors of $_{j}P_{j}$ and form the diagonal matrix $_{p_1}P_{p_1}$ containing all nonzero roots, or non-almost-zero roots, and the matrix $_{j}B_{p_1}$ containing the corresponding unit length characteristic vectors. A truncation of the characteristic roots and vector mode is envisaged with the number of roots retained in this step N_{p_1}, between the number of all nonzero roots, N_{p_2}, and the number of roots retained in the final approximation, N_p. It is important to discard only the very small roots at this time in addition to the zero roots. Elimination of very small roots, however, does aid in reduction of computations in subsequent steps and should not materially affect the results of these steps which assume that all nonzero roots are retained at this point.

3) Compute the matrix

(54) $$_{p_1}X_{(ik)} = {}_{p_1}B_{j}X_{(ik)} \,.$$

4) Compute the product matrix $_{k}Q_{k}$ by (46c) or by

(55) $$_{k}Q_{k} = {}_{k}X_{(ip_1)}X_{k} \,.$$

This latter procedure yields an approximation when almost-zero roots are discarded in the preceding step.

5) Determine the characteristic roots and vectors of $_{k}Q_{k}$ and form the diagonal matrix $_{q_1}Q_{q_1}$ containing all nonzero, or non-almost-zero roots and the matrix $_{k}C_{q_1}$ containing the corresponding unit-length characteristic vectors. As before, a truncation of the roots is envisaged between all nonzero roots retained at this stage and the roots retained for the final approximation.

6) Compute the matrix

(56) $$_{q_1}X_{(ip_1)} = {}_{q_1}C_{k}X_{(ip_1)} \,.$$

7) Rewrite ${}_{q_1}X_{(ip_1)}$ as ${}_iX_{(p_1q_1)}$ and compute the matrix

$$ {}_{(p_1q_1)}S_{(p_1q_1)} = {}_{(p_1q_1)}X_i X_{(p_1q_1)} \; . \tag{57}$$

Note that

$$ {}_iX_{(p_2q_2)} = {}_iA_{m_2}G_{(p_2q_2)} \; , \tag{58}$$

then

$$ {}_{(p_2q_2)}S_{(p_2q_2)} = {}_{(p_2q_2)}G_{m_2}A_i A_{m_2}G_{(p_2q_2)} = {}_{(p_2q_2)}G_{m_2}G_{(p_2q_2)} \; , \tag{59}$$

since ${}_iA_{m_2}$ is defined as a column-wise section of an orthonormal matrix. A consequence of (59) is that ${}_{(p_2q_2)}G_{m_2}$ is a factor matrix of ${}_{(p_2q_2)}S_{(p_2q_2)}$. The ${}_{(p_1q_1)}S_{(p_1q_1)}$ computed in (57) is taken as a section of ${}_{(p_2q_2)}S_{(p_2q_2)}$ obtained by deletion of rows and columns containing almost-zero entries.

8) Compute the characteristic roots and vectors of ${}_{(p_1q_1)}S_{(p_1q_1)}$ and form the diagonal matrix ${}_{m_1}S_{m_1}$ containing the nonzero or non-almost-zero roots and the matrix ${}_{(p_1q_1)}V_m$ containing the corresponding unit-length vectors.

9) Compute the matrices

$$ {}_{(p_1q_1)}G_{m_1} = {}_{(p_1q_1)}V_{m_1}S_{m_1}^{1/2} \tag{60}$$

and

$$ {}_iA_{m_1} = {}_iX_{(p_1q_1)}V_{m_1}S_{m_1}^{-1/2} \; . \tag{61}$$

Note that steps (8) and (9) utilize the Eckart and Young development [3] for approximation of the matrix ${}_iX_{(p_1q_1)}$ by the product of the matrices ${}_iA_{m_1}$ and ${}_{m_1}G_{(p_1q_1)}$. This development permits the analysis of a matrix utilizing a product matrix of order of the smaller mode of a two-mode matrix. This is the particular feature of Method II that permits analysis of data involving a large mode i.

10) Reduce the number of elements in each derivational mode to those elements that will be used in the approximation. Comments on this reduction are made in step (3) for Method I.

Method III

This method, like Method II, solves the problem of computations when the mode i for individuals in the sample is very large. Method III has some very interesting relations to the multitrait-multimethod matrix of Campbell and Fiske [2]. Furthermore, it is basic to the analysis allowing for measurement error type of uniqueness described in the next section of the report.

A matrix similar to the Campbell and Fiske multitrait-multimethod matrix may be defined as ${}_{(jk)}R_{(jk)}$. It could contain the correlations among the combination variables, but it is not restricted in this fashion. Whether the entries are correlations or not depends on the scaling of the entries in the matrix X. If the scaling for each combination variable is to standard scores divided by the square root of the number of individuals, this matrix would contain correlations; otherwise the entries would not be correlations. The definition of this matrix is

$$(62) \qquad {}_{(jk)}R_{(jk)} = {}_{(jk)}X_i X_{(jk)} .$$

It is a product matrix of the observed data matrix X and has rows and columns for the combination variables. The relation to the multitrait-multimethod matrix is more apparent if submatrices are defined of the form ${}_kR_k$. These are correlations among elements of mode k for specific values of mode j. Remember that mode j formed the outer loop for the combination mode (jk) and that mode k formed the inner loop for the combination mode (jk). The sectioned matrix may be represented as follows and is illustrated in Table 2.

$$\begin{bmatrix} {}_{1_j,k}R_{k,1_j} & {}_{1_j,k}R_{k,2_j} & {}_{1_j,k}R_{k,3_j} & \cdots \\ {}_{2_j,k}R_{k,1_j} & {}_{2_j,k}R_{k,2_j} & {}_{2_j,k}R_{k,3_j} & \cdots \\ {}_{3_j,k}R_{k,1_j} & {}_{3_j,k}R_{k,2_j} & {}_{3_j,k}R_{k,3_j} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The general form of the sections is ${}_{j,k}R_{k,j'}$ where j is used as a variable index for the elements in mode j and j' is used as an alternate variable index for the elements in mode j. The entries in the matrix ${}_{(jk)}R_{(jk)}$ are given by

$$(63) \qquad r_{jkj'k'} = \sum_i x_{ijk} x_{ij'k'} .$$

The entries in section ${}_{j,k}R_{k,j'}$ are for the specified values of j and j' for the specified section.

The entries in the product matrices ${}_jP_j$ and ${}_kQ_k$ have simple relations to the entries in the matrix ${}_{(jk)}R_{(jk)}$. From (46b)

$$(64) \qquad p_{jj'} = \sum_k \sum_i x_{ijk} x_{ij'k} .$$

Substitution from (63) yields

$$(65) \qquad p_{jj'} = \sum_k r_{jkj'k} .$$

Note that the entries $r_{jkj'k}$ for specific values of j and j' are diagonal entries in section ${}_{j,k}R_{k,j'}$. Thus, the value of $p_{jj'}$ is the sum of these diagonal entries which is the trace of this section.

$$(66) \qquad p_{jj'} = \mathrm{tr}\,({}_{j,k}R_{k,j'}).$$

The diagonal entries in the matrix ${}_jP_j$ are the traces of the corresponding diagonal sections in the matrix ${}_{(jk)}R_{(jk)}$ and the off-diagonal entries in ${}_jP_j$ are traces of corresponding off-diagonal sections of ${}_{(jk)}R_{(jk)}$.

The entries in the matrix ${}_kQ_k$ are, according to (46c),

(67) $$q_{kk'} = \sum_j \sum_i x_{ijk}x_{ijk'} .$$

Substitution from (63) yields

(68) $$q_{kk'} = \sum_j r_{jkjk'} .$$

Note that the entries $r_{jkjk'}$ occur only in the diagonal section of ${}_{(jk)}R_{(jk)}$ and that the summation in (68) for specific values of k and k' is over entries in corresponding locations in the diagonal sections of ${}_{(jk)}R_{(jk)}$. Thus, the matrix ${}_kQ_k$ can be expressed as the sum of the diagonal sections of the matrix ${}_{(jk)}R_{(jk)}$.

(69) $${}_kQ_k = \sum_j {}_{j,k}R_{k,j} .$$

Method III uses the foregoing relations to develop the matrices ${}_jP_j$ and ${}_kQ_k$. Following are the steps of Method III.

1) Compute the matrix ${}_{(jk)}R_{(jk)}$ by (62).

2) Compute the product matrices ${}_jP_j$ and ${}_kQ_k$ by (66) and (69).

3) Determine the characteristic roots and vectors of these product matrices and form the diagonal matrices of roots ${}_{p_1}P_{p_1}$ and ${}_{q_1}Q_{q_1}$ and matrices of vectors ${}_jB_{p_1}$ and ${}_kC_{q_1}$. All nonzero, or non-almost-zero roots are retained as in steps (2) and (5) for Method II.

4) Compute the matrix ${}_{(p_1q_1)}S_{(p_1q_1)}$, defined by (57), by

(70) $${}_{(p_1q_1)}S_{(p_1q_1)} = ({}_{p_1}B_j \times {}_{q_1}C_k)_{(jk)}R_{(jk)}({}_jB_{p_1} \times {}_kC_{q_1}).$$

5) Determine the characteristic roots and vectors of ${}_{(p_1q_1)}S_{(p_1q_1)}$ and form the diagonal matrix ${}_{m_1}S_{m_1}$ containing the nonzero, or non-almost-zero roots and the matrix ${}_{(p_1q_1)}V_{m_1}$ containing the corresponding unit-length vectors.

6) Compute the core matrix G by (60), repeated here for the readers' convenience.

$${}_{(p_1q_1)}G_{m_1} = {}_{(p_1q_1)}V_{m_1}S_{m_1}^{1/2} .$$

7) Compute the matrix ${}_iA_m$ by

(71) $${}_iA_{m_1} = {}_iX_{(jk)}({}_jB_{p_1} \times {}_kC_{q_1})_{(p_1q_1)}V_{m_1}S_{m_1}^{-1/2} .$$

8) Reduce the number of elements in each derivational mode to those elements that will be used in the approximation. Comments on this reduction are made in step (3) for Method I.

Model and Analysis with Unique Variance for Combination Variables

In the preceding section, the three-mode model has been used as a direct approximation to the observed data. This is more analogous to principal components analysis than to multiple-factor analysis which includes the concept of unique factors. A complete analogy to multiple-factor analysis has not been achieved for the three-mode model. However, a step in this direction has been made by the development to be described in this section in which the scores on combination variables are conceived of as arising from two sources, one being the scores generated by the three-mode model and the second being scores on variables unique to each of the combination variables. This revised model is described more precisely in the following paragraphs.

An assumption made in the revised model is that the sample of individuals is very large. This sample of individuals will be taken as mode i of the data. For defined conditions to be described in (73) and (74), following, to hold precisely, the sample size has to be unrestrictedly large. These relations probably will be adequately approximated with large-sample data.

Consider

$$\hat{x}_{ijk} = \tilde{x}_{ijk} + \ddot{x}_{ijk} , \tag{72}$$

where $\hat{x}_{ijk}$ is the approximation in (16) to the observed measures x_{ijk} , $\tilde{x}_{ijk}$ is that portion of the approximation defined by (17) and involves the three-mode model, and $\ddot{x}_{ijk}$ is that portion of the approximation that is unique to each combination variable. Let the three types of entries indicated above be entries in three-mode matrices $\hat{X}$, $\tilde{X}$, and $\ddot{X}$. Important properties of the unique portion for combination variables are defined in (73) and (74).

$$_{(jk)}\tilde{X}_i\ddot{X}_{(jk)} = 0, \tag{73}$$

$$_{(jk)}\ddot{X}_i\ddot{X}_{(jk)} = {}_{(jk)}U^2_{(jk)} , \tag{74}$$

where $_{(jk)}U^2_{(jk)}$ is a diagonal matrix containing entries $u^2_{(jk)}$ in the diagonal cells.

Tables 3 and 4 give the unique scores matrix $_i\ddot{X}_{(jk)}$ and the common score matrix $_i\tilde{X}_{(jk)}$ for the observed scores given in Table 1 for a fictitious body of data to be used to illustrate the relations discussed in this section. These data are set up so that there are no errors of approximation e_{ijk} of (16); and thus, the observed scores x_{ijk} in Table 1 equal the approximation scores $\hat{x}_{ijk}$ which are obtained by adding the unique scores $\ddot{x}_{ijk}$ in Table 3 to the common scores $\tilde{x}_{ijk}$ in Table 4. Each column of scores in matrix $_i\ddot{X}_{(jk)}$ in Table 3 is orthogonal to every other column of scores in this matrix, thus satisfying (74). Further, every column in $_i\ddot{X}_{(jk)}$ is orthogonal to every column in matrix $_i\tilde{X}_{(jk)}$ of Table 4, thus, satisfying (73). The sums of squares of the entries in the columns of $_i\ddot{X}_{(jk)}$, after being divided by N_i to convert them to mean squares, are listed in Table 2 in the row labeled $u^2_{(jk)}$ which is the next to last row of this table.

The matrix $_{(jk)}R_{(jk)}$ has been defined in (62). Correspondingly, let

TABLE 1

Observed Score Matrix ${}_iX_{(jk)}$ for Fictitious Problem

	1_j					2_j					3_j					4_j				
	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k
	$1_{(jk)}$	$2_{(jk)}$	$3_{(jk)}$	$4_{(jk)}$	$5_{(jk)}$	$6_{(jk)}$	$7_{(jk)}$	$8_{(jk)}$	$9_{(jk)}$	$10_{(jk)}$	$11_{(jk)}$	$12_{(jk)}$	$13_{(jk)}$	$14_{(jk)}$	$15_{(jk)}$	$16_{(jk)}$	$17_{(jk)}$	$18_{(jk)}$	$19_{(jk)}$	$20_{(jk)}$
1_i	46	33	7	10	9	35	27	17	12	13	43	36	28	15	20	35	37	36	20	26
2_i	-34	-25	-9	-26	-9	-17	5	-1	-12	17	1	0	12	7	12	1	11	48	4	22
3_i	10	-9	1	18	-9	-17	-15	1	-8	-13	-25	-12	-24	-11	-4	-19	-37	-36	-4	-26
4_i	-22	1	1	-2	9	-1	-17	-17	8	-17	-19	-24	-16	-11	-28	-17	-11	-48	-20	-22
5_i	26	33	7	-10	9	13	7	13	0	-9	31	8	20	23	-12	17	-7	12	32	-14
6_i	-14	-25	-9	-6	-9	-7	-15	-5	8	-5	-11	-4	4	15	12	-17	7	24	16	14
7_i	-10	-9	1	-2	-9	5	5	5	4	9	-13	-8	-16	-19	-4	-1	7	-12	-16	14
8_i	-2	1	1	18	9	-11	3	-13	-12	5	-7	4	-8	-19	4	1	-7	-24	-32	-14
9_i	46	15	7	10	-9	35	15	17	12	-1	29	24	28	1	4	19	17	36	4	10
10_i	-34	-7	-9	-26	9	-17	-7	-1	-12	3	-13	12	12	-7	28	17	31	48	20	38
11_i	10	9	1	18	9	-17	-3	1	-8	1	-11	-24	-24	3	-20	-35	-17	-36	-20	-10
12_i	-22	-17	1	-2	-9	-1	-5	-17	8	-3	-5	-12	-16	3	-12	-1	-31	-48	-4	-38
13_i	26	15	7	-10	-9	13	19	13	0	5	17	20	20	9	4	1	13	12	16	2
14_i	-14	-7	-9	-6	9	-7	-3	-5	8	9	-25	-16	4	1	-4	-1	-13	24	32	-2
15_i	-10	9	1	-2	9	5	-7	5	4	-5	1	4	-16	-5	12	-17	-13	-12	-32	-2
16_i	-2	-17	1	18	-9	-11	-9	-13	-12	-9	7	-8	-8	-5	-12	17	13	-24	-16	2
17_i	46	33	-7	-10	-9	35	27	3	-4	-1	43	36	12	1	4	35	37	24	4	10
18_i	-34	-25	-23	-6	9	-17	5	13	4	3	1	0	28	-7	28	1	11	36	20	38
19_i	10	-9	15	-2	9	-17	-15	-13	8	1	-25	-12	-8	3	-20	-19	-37	-24	-20	-10
20_i	-22	1	15	18	-9	-1	-17	-3	-8	-3	-19	-24	-32	3	-12	-17	-11	-36	-4	-38
21_i	26	33	-7	10	-9	13	7	-1	16	5	31	8	4	9	4	17	-7	24	16	2
22_i	-14	-25	-23	-26	9	-7	-15	9	-8	9	-11	-4	20	1	-4	-17	7	36	32	-2
23_i	-10	-9	15	18	9	5	5	-9	-12	-5	-13	-8	0	-5	12	-1	7	-24	-32	-2
24_i	-2	1	15	-2	-9	-11	3	1	4	-9	-7	4	-24	-5	-12	1	-7	-36	-16	2
25_i	46	15	-7	-10	9	35	15	3	-4	13	29	24	12	15	20	19	17	24	20	26
26_i	-34	-7	-23	-6	-9	-17	-7	13	4	17	-13	12	28	7	12	17	31	36	4	22
27_i	10	9	15	-2	-9	-17	-3	-13	8	-13	-11	-24	-8	-11	-4	-35	-17	-24	-4	-26
28_i	-22	-17	15	18	9	-1	-5	-3	-8	-17	-5	-12	-32	-11	-28	-1	-31	-36	-20	-22
29_i	26	15	-7	10	9	13	19	-1	16	-9	17	20	4	23	-12	1	13	24	32	-14
30_i	-14	-7	-23	-26	-9	-7	-3	9	-8	-5	-25	-16	20	15	12	-1	-13	36	16	14
31_i	-10	9	15	18	-9	5	-7	-9	-12	9	1	4	0	-19	-4	-17	-13	-24	-16	14
32_i	-2	-17	15	-2	9	-11	-9	1	4	5	7	-8	-24	-19	4	17	13	-36	-32	-14

$$(75) \qquad {}_{(jk)}\hat{R}_{(jk)} = {}_{(jk)}\hat{X}_i\hat{X}_{(jk)}$$

and

$$(76) \qquad {}_{(jk)}\tilde{R}_{(jk)} = {}_{(jk)}\tilde{X}_i\tilde{X}_{(jk)} \, .$$

It can be shown by substitution of a matrix form of (72) into (75) and algebraic simplification involving the definitions in (73), (74), and (76) that

$$(77) \qquad {}_{(jk)}\tilde{R}_{(jk)} = {}_{(jk)}\hat{R}_{(jk)} - {}_{(jk)}U^2_{(jk)} \, .$$

TABLE 2*

Matrix ${}_{(jk)}R_{(jk)}$

			1_j					2_j					3_j					4_j				
			1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k
			$1_{(jk)}$	$2_{(jk)}$	$3_{(jk)}$	$4_{(jk)}$	$5_{(jk)}$	$6_{(jk)}$	$7_{(jk)}$	$8_{(jk)}$	$9_{(jk)}$	$10_{(jk)}$	$11_{(jk)}$	$12_{(jk)}$	$13_{(jk)}$	$14_{(jk)}$	$15_{(jk)}$	$16_{(jk)}$	$17_{(jk)}$	$18_{(jk)}$	$19_{(jk)}$	$20_{(jk)}$
	1_k	$1_{(jk)}$	604	336	72	72	0	306	204	72	72	0	360	240	108	108	0	162	108	108	108	0
	2_k	$2_{(jk)}$	336	305	48	48	0	204	136	48	48	0	240	160	72	72	0	108	72	72	72	0
1_j	3_k	$3_{(jk)}$	72	48	145	96	0	24	-4	-40	0	-40	12	-32	-128	-48	-80	-36	-84	-264	-144	-120
	4_k	$4_{(jk)}$	72	48	96	196	0	24	-4	-40	0	-40	12	-32	-128	-48	-80	-36	-84	-264	-144	-120
	5_k	$5_{(jk)}$	0	0	0	0	81	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1_k	$6_{(jk)}$	306	204	24	24	0	271	147	60	42	18	261	192	108	72	36	162	135	144	90	54
	2_k	$7_{(jk)}$	204	136	-4	-4	0	147	149	56	26	30	192	158	114	54	60	135	135	174	84	90
2_j	3_k	$8_{(jk)}$	72	48	-40	-40	0	60	56	93	12	32	84	88	108	44	64	72	96	192	96	96
	4_k	$9_{(jk)}$	72	48	0	0	0	42	26	12	80	-4	48	28	24	32	-8	18	6	36	48	-12
	5_k	$10_{(jk)}$	0	0	-40	-40	0	18	30	32	-4	85	36	60	84	12	72	54	90	156	48	108
	1_k	$11_{(jk)}$	360	240	12	12	0	261	192	84	48	36	391	264	162	90	72	243	216	234	126	108
	2_k	$12_{(jk)}$	240	160	-32	-32	0	192	158	88	28	60	264	272	192	72	120	216	234	312	132	180
3_j	3_k	$13_{(jk)}$	108	72	-128	-128	0	108	114	108	24	84	162	192	344	112	168	162	234	516	264	252
	4_k	$14_{(jk)}$	108	72	-48	-48	0	72	54	44	32	12	90	72	112	137	24	54	54	204	168	36
	5_k	$15_{(jk)}$	0	0	-80	-80	0	36	60	64	-8	72	72	120	168	24	208	108	180	312	96	216
	1_k	$16_{(jk)}$	162	108	-36	-36	0	162	135	72	18	54	243	216	162	54	108	307	243	270	108	162
	2_k	$17_{(jk)}$	108	72	-84	-84	0	135	135	96	6	90	216	234	234	54	180	243	397	414	144	270
4_j	3_k	$18_{(jk)}$	108	72	-264	-264	0	144	174	192	36	156	234	312	516	204	312	270	414	1008	504	468
	4_k	$19_{(jk)}$	108	72	-144	-144	0	90	84	96	48	48	126	132	264	168	96	108	144	504	424	144
	5_k	$20_{(jk)}$	0	0	-120	-120	0	54	90	96	-12	108	108	180	252	36	216	162	270	468	144	388
Unique Mean Squares $u^2_{(jk)}$			100	81	49	100	81	64	36	49	64	49	49	36	64	49	64	64	100	36	64	64
Common Portion Mean Squares $r_{(jk)(jk)}$			504	224	96	96	0	207	113	44	16	36	342	236	280	88	144	243	297	972	360	324

*The entries in this table are mean products, the sums of products having been divided by N_i.

Since ${}_{(jk)}U^2_{(jk)}$ is a diagonal matrix by definition, only the diagonal entries of ${}_{(jk)}\hat{R}_{(jk)}$ are affected in obtaining the matrix ${}_{(jk)}\tilde{R}_{(jk)}$. This is analogous to insertion of communalities in the diagonal cells of correlation matrices in two-mode factor analysis.

Table 2 gives the matrix ${}_{(jk)}R_{(jk)}$ for the illustrative example. The entries in this matrix have been scaled to mean products which implies that the entries in the data matrix had been divided by the square root of the number of individuals in the sample. Such rescaling by a constant of proportionality has trivial effects on the model and the analysis, but should be remembered when relations with scores of individuals are considered. The rescaling to mean products was performed in order to obtain coefficients

TABLE 3

Unique Score Matrix ${}_i\ddot{X}_{(jk)}$

	1_j					2_j					3_j					4_j				
	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k
	$1_{(jk)}$	$2_{(jk)}$	$3_{(jk)}$	$4_{(jk)}$	$5_{(jk)}$	$6_{(jk)}$	$7_{(jk)}$	$8_{(jk)}$	$9_{(jk)}$	$10_{(jk)}$	$11_{(jk)}$	$12_{(jk)}$	$13_{(jk)}$	$14_{(jk)}$	$15_{(jk)}$	$16_{(jk)}$	$17_{(jk)}$	$18_{(jk)}$	$19_{(jk)}$	$20_{(jk)}$
1_i	10	9	7	10	9	8	6	7	8	7	7	6	8	7	8	8	10	6	8	8
2_i	-10	-9	7	-10	-9	-8	6	-7	-8	7	7	-6	-8	7	-8	-8	-10	6	-8	-8
3_i	10	-9	-7	10	-9	-8	-6	7	-8	-7	-7	6	-8	-7	8	8	-10	-6	8	-8
4_i	-10	9	-7	-10	9	8	-6	-7	8	-7	-7	-6	8	-7	-8	-8	10	-6	-8	8
5_i	-10	9	7	-10	9	-8	-6	7	-8	-7	7	-6	8	7	-8	8	-10	-6	8	-8
6_i	10	-9	7	10	-9	8	-6	-7	8	-7	7	6	-8	7	8	-8	10	-6	-8	8
7_i	-10	-9	-7	-10	-9	8	6	7	8	7	-7	-6	-8	-7	-8	8	10	6	8	8
8_i	10	9	-7	10	9	-8	6	-7	-8	7	-7	6	8	-7	8	-8	-10	6	-8	-8
9_i	10	-9	7	10	-9	8	-6	7	8	-7	-7	-6	8	-7	-8	-8	-10	6	-8	-8
10_i	-10	9	7	-10	9	-8	-6	-7	-8	-7	-7	6	-8	-7	8	8	10	6	8	8
11_i	10	9	-7	10	9	-8	6	7	-8	7	7	-6	-8	7	-8	-8	10	-6	-8	8
12_i	-10	-9	-7	-10	-9	8	6	-7	8	7	7	6	8	7	8	8	-10	-6	8	-8
13_i	-10	-9	7	-10	-9	-8	6	7	-8	7	-7	6	8	-7	8	-8	10	-6	-8	8
14_i	10	9	7	10	9	8	6	-7	8	7	-7	-6	-8	-7	-8	8	-10	-6	8	-8
15_i	-10	9	-7	-10	9	8	-6	7	8	-7	7	6	-8	7	8	-8	-10	6	-8	-8
16_i	10	-9	-7	10	-9	-8	-6	-7	-8	-7	7	-6	8	7	-8	8	10	6	8	8
17_i	10	9	-7	-10	-9	8	6	-7	-8	-7	7	6	-8	-7	-8	8	10	-6	-8	-8
18_i	-10	-9	-7	10	9	-8	6	7	8	-7	7	-6	8	-7	8	-8	-10	-6	8	8
19_i	10	-9	7	-10	9	-8	-6	-7	8	7	-7	6	8	7	-8	8	-10	6	-8	8
20_i	-10	9	7	10	-9	8	-6	7	-8	7	-7	-6	-8	7	8	-8	10	6	8	-8
21_i	-10	9	-7	10	-9	-8	-6	-7	8	7	7	-6	-8	-7	8	8	-10	6	-8	8
22_i	10	-9	-7	-10	9	8	-6	7	-8	7	7	6	8	-7	-8	-8	10	6	8	-8
23_i	-10	-9	7	10	9	8	6	-7	-8	-7	-7	-6	8	7	8	8	10	-6	-8	-8
24_i	10	9	7	-10	-9	-8	6	7	8	-7	-7	6	-8	7	-8	-8	-10	-6	8	8
25_i	10	-9	-7	-10	9	8	-6	-7	-8	7	-7	-6	-8	7	8	-8	-10	-6	8	8
26_i	-10	9	-7	10	-9	-8	-6	7	8	7	-7	6	8	7	-8	8	10	-6	-8	-8
27_i	10	9	7	-10	-9	-8	6	-7	8	-7	7	-6	8	-7	8	-8	10	6	8	-8
28_i	-10	-9	7	10	9	8	6	7	-8	-7	7	6	-8	-7	-8	8	-10	6	-8	8
29_i	-10	-9	-7	10	9	-8	6	-7	8	-7	-7	6	-8	7	-8	-8	10	6	8	-8
30_i	10	9	-7	-10	-9	8	6	7	-8	-7	-7	-6	8	7	8	8	-10	6	-8	8
31_i	-10	9	7	10	-9	8	-6	-7	-8	7	7	6	8	-7	-8	-8	-10	-6	8	8
32_i	10	-9	7	-10	9	-8	-6	7	8	7	7	-6	-8	-7	8	8	10	-6	-8	-8

unbiased by sample size. Due to the equivalence of the matrix X and matrix $\hat{X}$ for the example, the matrix in Table 2 can be considered also to be ${}_{(ik)}\hat{R}_{(ik)}$. As noted previously, the next to last row of Table 2 gives the diagonal entries $u^2_{(ik)}$ of matrix ${}_{(ik)}U^2_{(ik)}$. The last row of Table 2 gives the diagonal entries $\tilde{r}_{(ik)(ik)}$ for the matrix ${}_{(ik)}\tilde{R}_{(ik)}$. The entries in the row $\tilde{r}_{(ik)(ik)}$ were obtained by subtracting the $u^2_{(ik)}$ from the diagonal entries in the body of Table 2. This operation is in accord with (77). Further, according to (77), if the entries in the row $\tilde{r}_{(ik)(ik)}$ were substituted for the diagonal entries in the body of the table, the result would be the matrix ${}_{(ik)}\tilde{R}_{(ik)}$.

TABLE 4

Common Score Matrix ${}_{i}\tilde{X}_{(jk)}$

	1_j					2_j					3_j					4_j				
	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k	1_k	2_k	3_k	4_k	5_k
	$1_{(jk)}$	$2_{(jk)}$	$3_{(jk)}$	$4_{(jk)}$	$5_{(jk)}$	$6_{(jk)}$	$7_{(jk)}$	$8_{(jk)}$	$9_{(jk)}$	$10_{(jk)}$	$11_{(jk)}$	$12_{(jk)}$	$13_{(jk)}$	$14_{(jk)}$	$15_{(jk)}$	$16_{(jk)}$	$17_{(jk)}$	$18_{(jk)}$	$19_{(jk)}$	$20_{(jk)}$
1_i	36	24	0	0	0	27	21	10	4	6	36	30	20	8	12	27	27	30	12	18
2_i	-24	-16	-16	-16	0	-9	-1	6	-4	10	-6	6	20	0	20	9	21	42	12	30
3_i	0	0	8	8	0	-9	-9	-6	0	-6	-18	-18	-16	-4	-12	-27	-27	-30	-12	-18
4_i	-12	-8	8	8	0	-9	-11	-10	0	-10	-12	-18	-24	-4	-20	-9	-21	-42	-12	-30
5_i	36	24	0	0	0	21	13	6	8	-2	24	14	12	16	-4	9	3	18	24	-6
6_i	-24	-16	-16	-16	0	-15	-9	2	0	2	-18	-10	12	8	4	-9	-3	30	24	6
7_i	0	0	8	8	0	-3	-1	-2	-4	2	-6	-2	-8	-12	4	-9	-3	-18	-24	6
8_i	-12	-8	8	8	0	-3	-3	-6	-4	-2	0	-2	-16	-12	-4	9	3	-30	-24	-6
9_i	36	24	0	0	0	27	21	10	4	6	36	30	20	8	12	27	27	30	12	18
10_i	-24	-16	-16	-16	0	-9	-1	6	-4	10	-6	6	20	0	20	9	21	42	12	30
11_i	0	0	8	8	0	-9	-9	-6	0	-6	-18	-18	-16	-4	-12	-27	-27	-30	-12	-18
12_i	-12	-8	8	8	0	-9	-11	-10	0	-10	-12	-18	-24	-4	-20	-9	-21	-42	-12	-30
13_i	36	24	0	0	0	21	13	6	8	-2	24	14	12	16	-4	9	3	18	24	-6
14_i	-24	-16	-16	-16	0	-15	-9	2	0	2	-18	-10	12	8	4	-9	-3	30	24	6
15_i	0	0	8	8	0	-3	-1	-2	-4	2	-6	-2	-8	-12	4	-9	-3	-18	-24	6
16_i	-12	-8	8	8	0	-3	-3	-6	-4	-2	0	-2	-16	-12	-4	9	3	-30	-24	-6
17_i	36	24	0	0	0	27	21	10	4	6	36	30	20	8	12	27	27	30	12	18
18_i	-24	-16	-16	-16	0	-9	-1	6	-4	10	-6	6	20	0	20	9	21	42	12	30
19_i	0	0	8	8	0	-9	-9	-6	0	-6	-18	-18	-16	-4	-12	-27	-27	-30	-12	-18
20_i	-12	-8	8	8	0	-9	-11	-10	0	-10	-12	-18	-24	-4	-20	-9	-21	-42	-12	-30
21_i	36	24	0	0	0	21	13	6	8	-2	24	14	12	16	-4	9	3	18	24	-6
22_i	-24	-16	-16	-16	0	-15	-9	2	0	2	-18	-10	12	8	4	-9	-3	30	24	6
23_i	0	0	8	8	0	-3	-1	-2	-4	2	-6	-2	-8	-12	4	-9	-3	-18	-24	6
24_i	-12	-8	8	8	0	-3	-3	-6	-4	-2	0	-2	-16	-12	-4	9	3	-30	-24	-6
25_i	36	24	0	0	0	27	21	10	4	6	36	30	20	8	12	27	27	30	12	18
26_i	-24	-16	-16	-16	0	-9	-1	6	-4	10	-6	6	20	0	20	9	21	42	12	30
27_i	0	0	8	8	0	-9	-9	-6	0	-6	-18	-18	-16	-4	-12	-27	-27	-30	-12	-18
28_i	-12	-8	8	8	0	-9	-11	-10	0	-10	-12	-18	-24	-4	-20	-9	-21	-42	-12	-30
29_i	36	24	0	0	0	21	13	6	8	-2	24	14	12	16	-4	9	3	18	24	-6
30_i	-24	-16	-16	-16	0	-15	-9	2	0	2	-18	-10	12	8	4	-9	-3	30	24	6
31_i	0	0	8	8	0	-3	-1	-2	-4	2	-6	-2	-8	-12	4	-9	-3	-18	-24	6
32_i	-12	-8	8	8	0	-3	-3	-6	-4	-2	0	-2	-16	-12	-4	9	3	-30	-24	-6

A relation of the matrix ${}_{(jk)}\tilde{R}_{(jk)}$ is important relative to the estimation of the entries $u^2_{(jk)}$. These entries are analogous to the unique factor variances in two-mode factor analysis. By the relation to be developed below, these two problems can be brought together. By (76) and (18a)

$$(78) \qquad {}_{(jk)}\tilde{R}_{(jk)} = ({}_jB_p \times {}_kC_q)_{(pq)}G_m A_i A_m G_{(pq)}({}_pB_j \times {}_qC_k).$$

Let the factors among individuals be orthogonal so that the matrix ${}_iA_m$ is a column-wise section of an orthonormal matrix. This is the equivalent for the present case to the definition of uncorrelated factors in two-mode factor analysis. Equation (79) states this definition. Also, let the matrix ${}_{(jk)}F_m$ be defined as in equation (80).

$$(79) \qquad {}_mA_jA_m = {}_mI_m \ .$$

$$(80) \qquad {}_{(jk)}F_m = ({}_jB_p \times {}_kC_q)_{(pq)}G_m .$$

Substitution from (79) and (80) into (78) yields

$$(81) \qquad {}_{(jk)}\tilde{R}_{(jk)} = {}_{(jk)}F_mF_{(jk)} \ .$$

This is the familiar factor equation for two-mode factor analysis, here applied to the analysis of combination variables (jk). As a consequence, the diagonal entries $\tilde{r}_{(jk)(jk)}$ are the communalities in this two-mode factor analysis for combination variables. The same problems and methods of estimation of communalities exist here as for two-mode factor analysis.

The factor matrix ${}_{(jk)}F_m$ for the illustrative example is given in Table 5. This is a principal-axes factor matrix obtained from the characteristic roots and vectors of the matrix ${}_{(jk)}\tilde{R}_{(jk)}$ obtained by substitution of the entries $\tilde{r}_{(jk)(jk)}$ into the diagonal cells of the matrix ${}_{(jk)}\tilde{R}_{(jk)}$ in Table 2.

TABLE 5

Factor Matrix ${}_{(jk)}F_m$ of Matrix ${}_{(jk)}\tilde{R}_{(jk)}$

			1_m	2_m	3_m	4_m
	1_k	$1_{(jk)}$	11.0721	18.7719	- 4.0095	- 3.5982
	2_k	$2_{(jk)}$	7.3814	12.5146	- 2.6730	- 2.3988
1_j	3_k	$3_{(jk)}$	- 5.8294	7.5341	2.0812	- .9612
	4_k	$4_{(jk)}$	- 5.8294	7.5341	2.0812	- .9612
	5_k	$5_{(jk)}$	0	0	0	0
	1_k	$6_{(jk)}$	9.6334	10.6813	- .0013	.3268
	2_k	$7_{(jk)}$	8.7716	5.8652	1.2069	- .4496
2_j	3_k	$8_{(jk)}$	6.5584	- .2699	- .2932	- .9103
	4_k	$9_{(jk)}$	1.8598	2.2414	- 2.7086	.4247
	5_k	$10_{(jk)}$	4.6986	2.5113	2.4155	- 1.3350
	1_k	$11_{(jk)}$	13.7308	11.9766	2.0022	2.4528
	2_k	$12_{(jk)}$	13.8524	5.4731	3.7503	.3002
3_j	3_k	$13_{(jk)}$	16.0315	- 4.3068	- 1.6269	- 1.3400
	4_k	$14_{(jk)}$	6.6343	.7158	- 6.4579	1.3300
	5_k	$15_{(jk)}$	9.3971	- 5.0226	4.8309	- 2.6700
	1_k	$16_{(jk)}$	12.2921	3.8860	6.0104	6.3779
	2_k	$17_{(jk)}$	15.2430	- 1.1763	7.6302	2.2494
4_j	3_k	$18_{(jk)}$	28.4193	-12.1108	- 4.0013	- 1.2890
	4_k	$19_{(jk)}$	14.3236	- 4.5769	-11.2477	2.7160
	5_k	$20_{(jk)}$	14.0957	- 7.5339	7.2464	- 4.0050
Roots			2848.0285	1240.9651	421.7354	111.2651

TABLE 6

Mode j Product Matrix ${}_j\tilde{P}_j$ and Characteristic Roots and Vectors

	Matrix ${}_j\tilde{P}_j$					Roots of ${}_j\tilde{P}_j$		Matrix ${}_jB_p$	
	1_j	2_j	3_j	4_j				1_p	2_p
1_j	920	402	344	-174	1_p	3404.0670	1_j	.0697	.8613
2_j	402	416	631	645	2_p	1217.9328	2_j	.2936	.3168
3_j	344	631	1090	1377			3_j	.5527	.2030
4_j	-174	645	1377	2196			4_j	.7768	-.3415

TABLE 7

Mode k Product Matrix ${}_k\tilde{Q}_k$ and Characteristic Roots and Vectors

	Matrix ${}_k\tilde{Q}_k$						Roots of ${}_k\tilde{Q}_k$		Matrix ${}_kC_q$		
	1_k	2_k	3_k	4_k	5_k				1_q	2_q	3_q
1_k	1296	990	564	312	252	1_q	3185.0782	1_k	.5132	.6530	.1852
2_k	990	870	710	290	420	2_q	1039.8484	2_k	.4852	.3131	-.2142
3_k	564	710	1392	724	668	3_q	397.0731	3_k	.5780	-.5615	.0057
4_k	312	290	724	560	164			4_k	.2919	-.3170	.6810
5_k	252	420	668	164	504			5_k	.2861	-.2445	-.6753

In case the matrices ${}_jB_p$ and ${}_kC_q$ are column-wise sections of orthonormal matrices, (80) may be solved for the core matrix G.

$$(82) \qquad {}_{(pq)}G_m = ({}_pB_j \times {}_qC_k)_{(jk)}F_m \,.$$

If communality estimates can be obtained that are appropriate for two-mode factor analysis of the combination variables as indicated in (81), these estimates may be substituted in the diagonal cells of matrix ${}_{(jk)}R_{(jk)}$ to obtain an estimate of the matrix ${}_{(jk)}\tilde{R}_{(jk)}$. Then, this matrix may be used in Method III of analysis described in the preceding section. This analysis would start at step (2).

Table 6 presents the matrix ${}_jP_j$ obtained by (66) and Table 7 presents the matrix ${}_kQ_k$ obtained by (69) for the illustrative example. The characteristic roots and vectors for these two matrices are given also in Tables 6 and 7. All nonzero roots, and only nonzero roots, have been dropped. Since there were two nonzero roots for matrix ${}_jP_j$, there are two elements in the derivational mode p and N_p is 2. There are three nonzero roots for matrix ${}_kQ_k$, thus there are three elements in derivational mode q and N_q is 3. The matrices of

TABLE 8

Kronecker Product $({}_jB_p \times {}_kC_q)$

			1_p			2_p		
			1_q	2_q	3_q	1_q	2_q	3_q
			1_{pq}	2_{pq}	3_{pq}	4_{pq}	5_{pq}	6_{pq}
	1_k	$1_{(jk)}$	.0358	.0455	.0129	.4420	.5624	.1595
	2_k	$2_{(jk)}$	.0338	.0218	-.0149	.4179	.2697	-.1845
1_j	3_k	$3_{(jk)}$	.0403	-.0391	.0004	.4978	-.4836	.0049
	4_k	$4_{(jk)}$	.0203	-.0221	.0475	.2514	-.2730	.5865
	5_k	$5_{(jk)}$	.0199	-.0170	-.0471	.2461	-.2106	-.5816
	1_k	$6_{(jk)}$	.1508	.1919	.0544	.1626	.2069	.0587
	2_k	$7_{(jk)}$	.1426	.0920	-.0629	.1537	.0992	-.0679
2_j	3_k	$8_{(jk)}$	.1698	-.1650	.0017	.1831	-.1779	.0018
	4_k	$9_{(jk)}$	.0858	-.0931	.2001	.0925	-.1004	.2157
	5_k	$10_{(jk)}$	.0841	-.0718	-.1984	.0906	-.0775	-.2139
	1_k	$11_{(jk)}$	.2836	.3609	.1024	.1042	.1326	.0376
	2_k	$12_{(jk)}$	.2682	.1731	-.1184	.0985	.0636	-.0435
3_j	3_k	$13_{(jk)}$	.3195	-.3103	.0032	.1173	-.1140	.0012
	4_k	$14_{(jk)}$	.1613	-.1752	.3764	.0593	-.0644	.1382
	5_k	$15_{(jk)}$	.1581	-.1351	-.3732	.0581	-.0496	-.1371
	1_k	$16_{(jk)}$	.3987	.5073	.1439	-.1753	-.2230	-.0632
	2_k	$17_{(jk)}$	.3769	.2432	-.1664	-.1657	-.1069	.0731
4_j	3_k	$18_{(jk)}$	.4490	-.4362	.0044	-.1974	.1918	-.0019
	4_k	$19_{(jk)}$	.2267	-.2462	.5290	-.0997	.1083	-.2326
	5_k	$20_{(jk)}$	.2222	-.1899	-.5246	-.0977	.0835	.2306

characteristic vectors are the coefficient matrices ${}_jB_p$ and ${}_kC_q$. These matrices are given to the right of Tables 6 and 7.

The Kronecker product $({}_jB_p \times {}_kC_q)$ is given in Table 8. It is to be noted that since the matrices ${}_jB_p$ nad ${}_kC_q$ were column-wise sections of orthonormal matrices, this Kronecker product matrix is also a column-wise section of an orthonormal matrix.

Table 9 gives the matrix ${}_{(pq)}S_{(pq)}$ for the example computed according to (70) from the estimate of ${}_{(jk)}\tilde{R}_{(jk)}$ obtained by substitution of estimated communalities into the diagonal cells of the observed ${}_{(jk)}R_{(jk)}$. This change from (70) as stated is to be noted. Table 9 also presents the characteristic roots and vectors of the matrix ${}_{(pq)}S_{(pq)}$. There are four nonzero roots so that N_m is 4. The characteristic vectors form the matrix ${}_{(pq)}V_m$. When columns of ${}_{(pq)}V_m$ are multiplied by the square roots of the corresponding characteristic roots, as indicated in (60), the core matrix ${}_{(pq)}G_m$ is obtained. This matrix is given on the right of Table 9.

The core matrix could be obtained from (82) also, in case the matrix ${}_{(jk)}\tilde{R}_{(jk)}$ had been factored to the matrix ${}_{(jk)}F_m$. It is to be noted that the roots given for this factoring in Table 5 are equal to the roots obtained for the matrix ${}_{(pq)}S_{(pq)}$ and given in Table 9.

Another analogy with two-mode factor analysis shows up at this point.

TABLE 9

Matrix ${}_{(pq)}S_{(pq)}$, Characteristic Roots ${}_mS_m$, Characteristic Vectors ${}_{(pq)}V_m$, and Core Matrix ${}_{(pq)}G_m$

Matrix ${}_{(pq)}S_{(pq)}$

			1_p			2_p		
			1_q	2_q	3_q	1_q	2_q	3_q
			$1_{(pq)}$	$2_{(pq)}$	$3_{(pq)}$	$4_{(pq)}$	$5_{(pq)}$	$6_{(pq)}$
1_p	1_q	$1_{(pq)}$	2489.31	-288.90	-163.46	17.42	840.82	-190.07
	2_q	$2_{(pq)}$	-288.90	583.43	22.30	461.70	-17.28	173.08
	3_q	$3_{(pq)}$	-163.46	22.30	331.31	170.32	108.39	-.14
2_p	1_q	$4_{(pq)}$	17.42	461.70	170.32	695.75	288.91	163.47
	2_q	$5_{(pq)}$	840.82	-17.28	108.39	288.91	456.41	-22.30
	3_q	$6_{(pq)}$	-190.07	173.08	-.14	163.47	-22.30	65.76

Characteristic Roots ${}_mS_m$

1_m	2848.0285
2_m	1240.9651
3_m	421.7354
4_m	111.2651

Characteristic Vectors ${}_{(pq)}V_m$

			1_m	2_m	3_m	4_m
1_p	1_q	$1_{(pq)}$	.9320	-.0085	.1605	.2052
	2_q	$2_{(pq)}$	-.1242	.5665	.5000	.5677
	3_q	$3_{(pq)}$	-.0464	.1850	-.7626	.5804
2_p	1_q	$4_{(pq)}$	.0158	.7344	-.1003	-.4400
	2_q	$5_{(pq)}$	.3290	.2692	-.3391	-.2943
	3_q	$6_{(pq)}$	-.0731	.1818	.1328	-.1364

Core Matrix ${}_{(pq)}G_m$

			1_m	2_m	3_m	4_m
1_p	1_q	$1_{(pq)}$	49.738	-.299	3.296	2.164
	2_q	$2_{(pq)}$	-6.628	19.956	10.268	5.988
	3_q	$3_{(pq)}$	-2.476	6.517	-15.661	6.122
2_p	1_q	$4_{(pq)}$	.843	25.871	-2.060	-4.641
	2_q	$5_{(pq)}$	17.558	9.483	-6.964	-3.104
	3_q	$6_{(pq)}$	-3.901	6.404	2.727	-1.439

The matrix $_iA_m$ is not determinate. This is precisely the same problem as indicated by considering (71) in step (7) of Method III factoring. The appropriate X matrix to use is an estimate of $\tilde{X}$. However, an estimate of neither $\tilde{X}$ nor $\ddot{X}$ is available. The definitions of (73) and (74) force the combination variables scores in these two matrices to be located in a space of higher dimensionality than the space for the original combination variable score. Thus, the column vectors of matrix $_iA_m$ are in this higher dimensional space and are not determinate from the data in matrix X for observed scores.

While the factor coefficients for individuals in matrix $_iA_m$ are not determinate, (71) may be used with the observed score matrix X to obtain a type of estimate of these factor coefficients. The properties of these estimates have not been explored. Consequently, these estimates should be used with considerable caution.

Once the coefficient matrices $_jB_p$ and $_kC_q$ and the core matrix $_{(pq)}G_m$ have been determined, the problem of transformation of derivational modes should be considered. Comments on this problem are given in association with (25) through (31). It is to be noted that no restrictions were made that the transformation matrices had to be orthonormal. It is not at all clear what this restriction would mean for modes p and q. For mode m, the meaning involves restrictions whether elements of this mode should be orthogonal in the matrix $_iA_m$ or $_iA_{m^*}$. The orthogonal restriction for this matrix is equivalent to the restriction to uncorrelated factors in two-mode factor analysis. Transformations on modes p and q in coefficient matrices $_jB_p$ and $_kC_q$ result in inverse transformation on the core matrix G but do not affect the matrix $_iA_m$. As previously indicated, attempted transformations on modes p and q

TABLE 10

Transformations of Derivational Modes

$_pT_{p^*}$

	1_{p^*}	2_{p^*}
1_p	.986	3.729
2_p	2.242	-.302

$_qT_{q^*}$

	1_{q^*}	2_{q^*}	3_{q^*}
1_q	2.510	1.740	2.214
2_q	2.585	-1.757	-1.299
3_q	.127	1.373	-1.554

$_mT_{m^*}$

	1_{m^*}	2_{m^*}	3_{m^*}	4_{m^*}
1_m	.906	.228	.128	.332
2_m	-.259	.658	.707	-.015
3_m	-.312	-.147	.043	.938
4_m	.117	-.702	.695	-.103

$_jB_{p^*}$

	1_{p^*}	2_{p^*}
1_j	2	0
2_j	1	1
3_j	1	2
4_j	0	3

$_kC_{q^*}$

	1_{q^*}	2_{q^*}	3_{q^*}
1_k	3	0	0
2_k	2	0	1
3_k	0	2	2
4_k	0	2	0
5_k	0	0	2

$_{(p^*q^*)}G_m$

			1_{m^*}	2_{m^*}	3_{m^*}	4_{m^*}
	1_{q^*}	$1_{(p^*q^*)}$	1	3	2	0
1_{p^*}	2_{q^*}	$2_{(p^*q^*)}$	-2	1	1	0
	3_{q^*}	$3_{(p^*q^*)}$	0	0	0	0
	1_{q^*}	$4_{(p^*q^*)}$	1	0	1	1
2_{p^*}	2_{q^*}	$5_{(p^*q^*)}$	3	0	0	-1
	3_{q^*}	$6_{(p^*q^*)}$	2	0	-1	2

to matrices ${}_jB_{p*}$ and ${}_kC_{q*}$ to simple structure for experimental studies have led to considerable success. These transformations have not been limited to orthonormal matrices but have involved oblique transformations. Transformation on the mode m have been problematic due to the indeterminancy of the matrix ${}_iA_m$ for the case with unique variance for the combination variables. Some success has been obtained by inverse transformations on the core matrix G to a simple structure.

Examples of the transformation of derivational modes for the illustrative problem are given in Table 10. In this case, the transformations were to obtain the matrices that were used to produce the example rather than to obtain a simple structure. The matrices ${}_jB_{p*}$, ${}_kC_{q*}$, and ${}_{(p*q*)}G_{m*}$ given in Table 10 are the matrices used in setting up the example. The matrices ${}_mT_{m*}$, ${}_pT_{p*}$, and ${}_qT_{q*}$ transform the matrices ${}_jB_p$, ${}_kC_q$, and ${}_{(pq)}G_m$ to the input matrices by (25b), (25c), and (29a).

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