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Theoretical Note

A Note on the Exact Number of Two- and Three-Way Tables Satisfying Conjoint Measurement and Additivity Axioms*

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A technique for enumerating and counting the number of two- and three-way tables which satisfy polynomial conjoint measurement test and additivity tests is presented. The counting technique reveals that the number of tables which satisfy all the constraints is very small relative to the number of possible tables. © 1993 Academic Press, Inc.

Arbuckle and Larimer (1976) presented techniques for estimating the number of two-way tables that satisfy the conjoint measurement axioms presented by Krantz and Tversky (1971) and derived the formula for the exact number of *regular* $r \times c$ tables

$$N_{\text{reg}}(r, c) = rc! \Big/ \prod_{i=2}^{r} \prod_{j=2}^{c} r!\, c!\, (i+j-1).$$

A two-way table is *regular*[1] if it satisfies Krantz and Tversky's independence condition and if the entries in the rows always increase from left to right and increase in the columns from top to bottom. Arbuckle and Larimer performed their estimates by using a monte carlo procedure that generated regular tables that were subsequently tested for double cancellation and additivity.

McCelland (1977) was able to determine the exact number of 3×3, 3×4, 3×5, and 4×4 two-way tables that satisfied the constraints of double cancellation and additivity. McCelland's technique was to enumerate all possible regular matrices of each of the above sizes followed by testing for double cancellation and additivity.

This paper presents the results for a larger set of two-way tables and for the $3\times3\times2$, $3\times4\times2$, and $3\times3\times3$ three-way tables that satisfy the conjoint measurement axioms. The exact number of $3\times3\times2$ tables is of particular interest because this is the smallest number of three dimensional data matrices for which it may be

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[1] McCelland's term for regular is *independent normal*.

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possible to distinguish between the additive law $A+P+U$, and the distributive law, $(A+P)U$.

A three-way regular $a \times p \times u$ table consists of a three dimensional array of the integers 1, 2, ..., *apu* in which the integers are always increasing from right to left, from top to bottom, and from front to rear. The computer program used to enumerate the number of three-way tables starts with an empty three dimensional array, then places the digit 1 in the uppermost, leftmost, frontmost corner. This is the root node at the top level of a tree (Knuth, 1973). The three nodes of the second level of the tree are formed by placing the digit 2 immediately to the left of, to the right of, or to the rear of the 1. The nodes at the third level of the tree are formed by placing the digit 3 to the left of, to the right of, and/or to the rear of either the 1 or the 2 in the level two nodes, subject to the restriction that there must not be an empty cell to the left of, above, or in front of the 3. This technique is continued until all empty cells are filled in, at which time the Ullrich and Wilson (1990) programs are applied. These programs test the conjoint measurement test from the Krantz and Tversky paper and the additivity test from the Sherman (1977) algorithm. To save computational time, the tree may be pruned by performing tests of double cancellation or joint independence before the three dimensional array is completely filled. A test of the axioms is considered successful if it satisfies the antecedent and the consequent conditions or if it fails to satisfy the antecedent conditions of the Krantz and Tversky tests. The number of two-way tables may be calculated using this same program with the number of levels of the third dimension equal to one.

The computers currently available are much faster than those available to McCelland a decade ago. Accordingly, his results were extended to a larger set. The results[2] for the two-way tables are presented in Table 1. Examination of this table clearly reveals that the proportion of additive matrices dramatically decreases with increasing size as measured by $r \times c$ or by the number of regular tables.

When the enumeration program[3] was used to calculate the number of three-way tables, it was found that there were 4,877,756 regular $3 \times 3 \times 2$ tables. All of these, by definition, satisfy the single cancellation test in all three dimensions. A total of 3,509,856 of these satisfy all the double cancellation tests in the $A \times P$ planes for the two levels of U. All tests mentioned subsequently in this paragraph pass these double cancellation and independence tests. When the joint independence tests are applied, 2,737,620 of these fail in all three dimensions ($A \times P{:}U$, $A \times U{:}P$, and $P \times U{:}A$). None of these tables satisfy the axioms for the additive, distributive, or dual distributive laws. A total of 767,344 tables satisfied the joint independence tests in exactly one or exactly two dimensions; these are candidates for the distributive or dual distributive laws. Only 4,796 satisfied the joint independence tests in all

[2] McCelland's entry for the 4×4 table contains an error in that his number 7840 should be the sum of 932 and 6660, or 7592.

[3] Two completely separate programs, one in the *C* programming language and one in the PROLOG programming language, were used to verify correctness of the 4,877,756 number.

TABLE 1

The Number of Two-Way Tables of Different Sizes Satisfying the Conjoint Measurement and Additivity Tests

	3	4	5
			Legend
3	362,880		P
	42		R & ~DC
	6		R & DC & ~JI
	0		R & DC & JI & ~A
	36		R & DC & JI & A
4	479,001,600	$2.09E+13$	
	462	24,024	
	159	16,432	
	8	932	
	295	6,660	
5	$1.31E+12$	$2.43E+18$	$1.55E+25$
	6,006	1,662,804	701,149,020
	3,233	1,459,657	?
	190	50,213	6,747,173
	2,583	152,934	84,499,471
6	$6.40E+15$	$6.20E+23$	?
	87,516	140,229,804	?
	60,784	?	?
	3,152	2,110,631	?
	23,580	3,533,829	?

Note. The legend contains mnemonics for the table entries: P represents possible, R regular, DC double cancellation, JI joint independence, and A additive. Entries with a question mark were not calculated.

three dimensions and of these 3,792 satisfied the strict test of additivity. Because so few matrices satisfied this condition, it is apparent that the constraints imposed by additivity are extremely strong. It should be noted that some of these data matrices may in fact be generated by distributive or dual-distributive laws (Emery and Barron, 1979; Nygren, 1985). A copy of the 3,792 additive data matrices may be obtained from the authors.

There does not appear to be any simple equation for calculating the exact number of regular three-way tables insofar as the prime factors of 4,877,756 are 2, 2, 13, 19, and 4937. The 4937 factor seems particularly intractable.

With respect to three-way tables larger than $3 \times 3 \times 2$, it was necessary to prune the tree as it was being created for failures of joint independence or of double cancellation in any of the $A \times P$ planes. This prevents the algorithm from counting the number of regular matrices. For the $3 \times 3 \times 3$ case there were 1,865,778 tables

TABLE 2

The Number of Three-Way Tables Satisfying the Conjoint Measurement and Additivity Tests

	$3 \times 3 \times 2$	$3 \times 4 \times 2$	$3 \times 3 \times 3$
P	6.40 *E*15	6.20 *E*23	1.08 *E*28
R	4,877,756	?	?
R & DC3	3,509,856	?	?
R & DC3 & JI0	2,737,620	?	?
R & DC3 & (JI1 | JI2)	767, 344	?	?
R & DC3 & J3	4,796	273,860	1,865,778
R & DC3 & J3 & *A*	3,792	141,344	566,616

Note. The mnemonics are the same as those for Table 1 with the addition of a digit indicating the number of test passed.

which satisfied all the joint independence and double cancellation tests. Of these, 566,616 were additive. In the $3 \times 4 \times 2$ case there were 273,860 tables which satisfied the joint independence and double cancellation tests. Of these, 141,344 tables were additive. Table 2 contains a summary of the tests for the $3 \times 3 \times 2$, $3 \times 4 \times 2$, and $3 \times 3 \times 3$ cases.

Judging from the relatively small number of tables which are additive, it is apparent that additivity is an extremely strong constraint. The ability to enumerate each and every additive matrix could be useful in specifying an attempt to construct an error theory for conjoint measurement. For example, the number of data transformations necessary to make a data table containing errors into an additive table could be determined. Both Coombs and Huang (1970) and Ullrich and Painter (1974) have used this technique. An unsolved problem, of course, remains as to the number of tables satisfying the distributive and dual-distributive laws. The solution to this problem is considerably more difficult because of the non-linear form of the constraints. Since the number of $3 \times 3 \times 2$ tables which satisfied joint independence in all three dimensions was 4,796 (a necessary condition for the additive law), and since there were 767,344 which satisfied joint independence in one or two dimensions (a necessary condition for distributive and dual-distributive laws), there is a strong suggestion that the distributive and dual-distributive laws are much weaker. Equivalently, there is the suggestion that there are far more distributive and dual-distributive tables than additive tables.

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